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QUESTIONS

IN

PURE MATHEMATICS

PROPOSED AT THE

B.A. AND B.SC. PASS AND HONOURS EXAMINATIONS

OF THE

UNIVERSITY OF LONDON.

WITH

COMPLETE SOLUTIONS

BY

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PREFACE.

THE main object of this book is to afford help to those students who have to read Mathematics for the B.A. and B.Sc. degrees of the University of London without the aid of a private tutor. Its appearance is due to the suggestion of my friend Mr. E. M. LANGLEY, Mathematical Master in the Bedford Modern School; and it was originally intended to be our joint production. The pressure of other duties has prevented Mr. LANGLEY from carrying out his part of the work; and though I am much indebted to him for his advice and criticism, he is not responsible for any thing in the present volume beyond his solution of Question 9, page 17, which I have substituted for my own.

In the various divisions of the book I have tried to adapt the methods employed to the standard that a student may be expected to have reached at the corresponding point of his, or her, course. It therefore happens, particularly in the earlier papers, that the solutions chosen are

not always the shortest, but such as, I venture to think, would either present the subject of the question most clearly to the student, or would supplement some better-known methods.

I have tried to reduce the number of books of reference, as far as possible, by choosing *one* in each subject for general use. It thus happens that a piece of "book work" which is to be found in some text-books, but not in those adopted throughout, has been occasionally embodied in the text of this volume.

I have to express my thanks to Mr. J. W. Capstick, B.A., for his careful correction of several of the earlier proof-sheets.

J. E. A. STEGGALL.

The Owens College, Manchester,
July 27, 1882.

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LIST OF BOOKS
REFERRED TO IN THIS WORK.

I. GENERAL REFERENCE.

TODHUNTER, I.
- Euclid.
- Algebra for Colleges and Schools.
- Plane Trigonometry.
- Spherical Trigonometry.
- Theory of Equations.

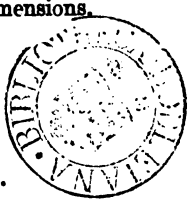
BESANT, W. H.
- Geometrical Conic Sections.

WILSON, J. M.
- Solid Geometry and Conic Sections.

SALMON, G.
- Higher Algebra.
- Conic Sections.
- Higher Plane Curves.
- Geometry of Three Dimensions.

WILLIAMSON, B.
- Differential Calculus.
- Integral Calculus.

BOOLE, G.
- Differential Equations.



II. OCCASIONAL REFERENCE.

DREW, W. H.
- Geometrical Conic Sections.

TODHUNTER, I.
- Differential Calculus.

BOOLE, G.
- Finite Differences.

CAYLEY, A.
- Elliptic Functions.

ERRATA.

Page 45, Qu. 11: *for* $P_1 Q_1 R_1$ *read* P, Q, R.

„ 160, line 3 from bottom: for $u - iy$ read $u - iv$.

In the names of EXAMINERS, pp. 106, 117, 124, 137:

for Rev. Prof. TOWNSEND, M.A., F.R.S.,

read BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

FIRST B.A. AND B.Sc. EXAMINATION.

PURE MATHEMATICS.

PASS EXAMINATION, 1877.

Wednesday, July 18, 1877.

Examiners, { Prof. HENRICI, Ph.D., F.R.S.
Rev. Prof. TOWNSEND, M.A., F.R.S.

Morning, 10 *to* 1.

ARITHMETIC AND ALGEBRA.

1. *Find to the nearest shilling, the present value of* £273, *payable after three years, the rate of interest being* 3 *per cent. per annum.*

The amount of £1 for 3 years at 3 per cent. compound interest (which we are clearly meant to consider) is £1·092727:

$\therefore$ present value of £1·092727 is £1,

$\therefore$ present value of £273 is $£\frac{273}{1\cdot092727}$.

We need only retain four decimal places in the denominator, since one shilling is greater than the 10,000th part of £273; dividing then by abridged division we obtain £249·84 or £249 17*s.*, the result required.

1·0927)273 (249·84
21854
5446
4371
1075
983
92
87
5

2. *Extract the square root of* $2+\sqrt{3}$, *and* $2-\sqrt{3}$.

Let $$\sqrt{2+\sqrt{3}}=\sqrt{x}+\sqrt{y}, \quad . \quad . \quad . \quad . \quad . \quad (a)$$

then $$\sqrt{2-\sqrt{3}}=\sqrt{x}-\sqrt{y}; \quad . \quad . \quad . \quad . \quad (b)$$

$$\therefore\ 2+\sqrt{3}=x+y+2\sqrt{xy}; \quad . \quad . \quad . \quad . \quad (c)$$

and $\sqrt{4-3}=1=x-y$, by multiplication of (a) and (b);

but $2 = x+y$, from (c),

$$\therefore\ x=\tfrac{3}{2},\quad y=\tfrac{1}{2};$$

$\therefore$ the roots are, respectively,

$$\sqrt{\tfrac{3}{2}}+\sqrt{\tfrac{1}{2}} \text{ and } \sqrt{\tfrac{3}{2}}-\sqrt{\tfrac{1}{2}}, \text{ or } \frac{\sqrt{3}\pm 1}{\sqrt{2}}.$$

3. *Having given* $\log 193{\cdot}06 = 2{\cdot}2856923$,

$\log 19307 = 4{\cdot}2857148$,

find the 7th root of 100 *to six decimals.*

$$\text{Log } \sqrt[7]{100}=\tfrac{1}{7}\log 100=\tfrac{2}{7}$$
$$=\cdot 2857143 \text{ to 7 figures};$$
$$\log 1{\cdot}9306=\cdot 2856923,$$
$$\log 1{\cdot}9307=\cdot 2857148;$$

$\therefore$ difference for $\cdot 0001$ is 225,

and given difference is 220;

$\therefore$ quantity to be added to 1·9306 is $\frac{220}{225}$ of $\cdot 0001$, or quantity to be subtracted from 1·9307 is $\frac{5}{225}$ of $\cdot 0001$, *i. e.* $\cdot 000002$;

$\therefore$ required root is 1·930698.

4. *If* $\kappa=x\sqrt{1+y^2}+y\sqrt{1+x^2}$,

prove that $\sqrt{1+\kappa^2}=xy+\sqrt{1+x^2}\,\sqrt{1+y^2}$.

Since $\kappa = x\sqrt{1+y^2}+y\sqrt{1+x^2}$,

$$\therefore\ \kappa^2=x^2(1+y^2)+y^2(1+x^2)+2xy\sqrt{(1+x^2)(1+y^2)};$$

$$\therefore\ \kappa^2+1=1+x^2+y^2+2x^2y^2+2xy\sqrt{(1+x^2)(1+y^2)}$$
$$=(1+x^2)(1+y^2)+2xy\sqrt{(1+x^2)(1+y^2)}+x^2y^2;$$
$$\therefore\ \sqrt{\kappa^2+1}=\sqrt{(1+x^2)(1+y^2)}+xy.$$

5. *If x, y, z be proportional to given quantities a, b, c, respectively, and if the sum of their squares be* 100, *find their values.*

Here $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}=\kappa$ say,

and $x^2+y^2+z^2=100,$

or $\kappa^2a^2+\kappa^2b^2+\kappa^2c^2=100,$

for $x=\kappa a,\quad y=\kappa b,\quad z=\kappa c,$

hence $\kappa=\dfrac{\pm 10}{\sqrt{a^2+b^2+c^2}};$

$$\therefore\ x=\pm\frac{10a}{\sqrt{a^2+b^2+c^2}},$$
$$y=\pm\frac{10b}{\sqrt{a^2+b^2+c^2}},$$
$$z=\pm\frac{10c}{\sqrt{a^2+b^2+c^2}},$$

where all the signs must be taken alike.

6. *Three persons have four coats, five vests, and six hats between them; in how many different ways can they dress themselves with them?*

Three coats can clearly be distributed amongst them in $4.3.2$, or 24 ways; three vests in $5.4.3$, or 60 ways; three hats in $6.5.4$, or 120 ways; and each arrangement of the coats can be combined with each arrangement of the vests, and with each arrangement of the hats: thus the whole number of ways in which the three persons may attire themselves, each with a coat, vest, and hat, is

$$24\times 60\times 120, \text{ or } 172800 \text{ ways.}$$

7. *Solve the equations:*

$$\begin{aligned} 3x+9y-\ 6z&=\ 8,\\ 4x-7y+13z&=\ 9,\\ 12x-3y\ -5z&=11. \end{aligned}$$

Eliminating x from the first and third equations and also from the second and third, we get readily

$$\begin{aligned} 39y-\ 19z&=21,\\ -18y+\ 44z&=16\\ \text{or}\qquad -\ 9y+\ 22z&=8.\\ \text{Hence}\qquad 117y-\ 57z&=63,\\ \text{and}\qquad -117y+286z&=104;\\ \therefore\ 229z\qquad &=167,\\ z\qquad &=\tfrac{167}{229},\\ y\qquad &=\tfrac{614}{687},\\ x\qquad &=1\tfrac{305}{687}. \end{aligned}$$

8. *Find all the solutions of the system of equations:—*

$$\begin{aligned} x^2+xy&=(a-b)^2,\\ xy+y^2&=4ab. \end{aligned}$$

Adding

$$\begin{aligned} (x+y)^2&=a^2+2ab+b^2\\ &=(a+b)^2,\\ \therefore\quad x+y&=\pm(a+b):\\ \text{but}\qquad x(x+y)&=(a-b)^2,\\ y(x+y)&=4ab; \end{aligned}$$

$$\therefore\ x=\pm\frac{(a-b)^2}{a+b},$$

$$y=\pm\frac{4ab}{a+b},$$

like signs being taken.

9. *Out of a cask containing* 360 *quarts of pure alcohol a quantity is drawn off and replaced by water. Of the mixture a second quantity,* 84 *quarts more than the first, is drawn off and replaced by water. The cask now contains as much water as alcohol. Find how many quarts were taken out the first time. Show that the problem has only one solution.*

Let x=the number of quarts drawn off at first; the vessel now contains $360-x$ quarts of alcohol and x of water; in the $x+84$ quarts which are next drawn off, there must therefore be

$$\frac{360-x}{360}\times(x+84) \text{ quarts of alcohol;}$$

$\therefore$ total amount of alcohol drawn off is

$$x+\frac{360-x}{360}(x+84) \text{ quarts;}$$

but this is half the contents, or 180 quarts,

$$\therefore\ x+\frac{360-x}{360}(x+84)=180,$$

or $$360x+30240+276x-x^2=64800;$$

$$\therefore\ x^2-636x+34560=0,$$

or $$(x-576)(x-60)=0.$$

The only admissible solution is $x=60$, the number of quarts at first drawn off. The other value of x, viz. 576, does not suit this question; for the cask only contained 360 quarts originally.

10. *Find the L.C.M., and the factors, of the expressions*

$$3x^3+x^2-8x+4,\quad 3x^3+7x^2-4,$$
$$x^3+2x^2-x-2,\quad 3x^3-2x^2-3x+2;$$

and also the factors of their sum. Find the sum of their reciprocals.

The common factors, and thence the others, may be readily found for any pair by the usual process for G.C.M.; but a very little consideration will give us the factors by inspection.

Thus, taking the third and fourth expressions,

$$x^3+2x^2-x-2=x^2(x+2)-(x+2)$$
$$=(x^2-1)(x+2)=(x-1)(x+1)(x+2),$$
$$3x^3-2x^2-3x+2=x^2(3x-2)-(3x-2)$$
$$=(x^2-1)(3x-2)=(x-1)(x+1)(3x-2);$$

these results suggest $x-1$, $x+1$, $3x-2$, and $x+2$ as trial factors for the other expressions: proceeding to them,

$$3x^3+x^2-8x+4=3x^3-3x^2+4x^2-4x-4x+4$$
$$=3x^2(x-1)+4x(x-1)-4(x-1)$$
$$=(3x^2+4x-4)(x-1)$$
$$=(3x-2)(x+2)(x-1).$$

Lastly

$$3x^3+7x^2-4=3x^3+3+7x^2-7$$
$$=3(x^3+1)+7(x^2-1)=(x+1)\{3(x^2-x+1)+7(x-1)\}$$
$$=(x+1)\{3x^2+4x-4\}=(x+1)(3x-2)(x+2).$$

Hence the L.C.M. is

$$(x-1)(x+1)(x+2)(3x-2)$$
$$=3x^4+4x^3-7x^2-4x+4. \quad \ldots \quad \text{I.}$$

The sum is $10x^3+8x^2-12x=2x(5x^2+4x-6)$,

which does not yield any further simple rational factors. II.

The sum of the reciprocals is

$$\frac{1}{(x-1)(x+1)(x+2)}+\frac{1}{(x-1)(x+1)(3x-2)}$$
$$+\frac{1}{(3x-2)(x+2)(x-1)}+\frac{1}{(3x-2)(x+2)(x+1)}$$
$$=\frac{3x-2+x+2+x+1+x-1}{(x-1)(x+1)(x+2)(3x-2)}$$
$$=\frac{6x}{(x-1)(x+1)(x+2)(3x-2)}. \quad \ldots \quad \text{III.}$$

11. *Convert the circulating decimal fraction* $1\cdot\dot{4}63\dot{1}$ *into a vulgar fraction. Find also the sum of*

$$\frac{1}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{3}{8^4}+\frac{1}{8^5}+\frac{4}{8^6}+\frac{6}{8^7}+\frac{3}{8^8}+\frac{1}{8^9}+\ldots \text{ to infinity,}$$

the numerators being 1, 4, 6, 3, *recurring.*

$$1\cdot\dot{4}63\dot{1}=10\times\cdot\dot{1}46\dot{3}=\frac{14630}{9999}$$
$$=\frac{1330}{909}=1\frac{421}{909}.$$

The series consists of the four series

$$\frac{1}{8}+\frac{1}{8^5}+\ldots \text{ to infinity,}$$

$$+\frac{4}{8^2}+\frac{4}{8^6}+\ldots \quad ,,$$

$$+\frac{6}{8^3}+\frac{6}{8^7}+\ldots \quad ,,$$

$$+\frac{3}{8^4}+\frac{3}{8^8}+\ldots \quad ,,$$

and the sum of these (by the formula for the sum of an infinite geometrical progression) is

$$\frac{\frac{1}{8}+\frac{4}{8^2}+\frac{6}{8^3}+\frac{3}{8^4}}{1-\frac{1}{8^4}}=\frac{8^3+4.8^2+6.8+3}{8^4-1}$$
$$=\frac{512+256+48+3}{4095}=\frac{819}{4095}=\frac{1}{5}.$$

This may be seen at once by observing that we require the value of the recurring decimal $\cdot\dot{1}46\dot{3}$ in scale 8, which equals $\frac{1463}{7777}$, all being expressed in scale 8; or

$$\frac{8^3+4.8^2+6.8+3}{8^4-1}, \text{ as before, } =\frac{1}{5}.$$

12. *If y denote the expression $ax+b$, show that the values which y assumes when values in arithmetical progression are substituted for x are themselves in arithmetical progression.*

Let the values

$$p,\ p+q,\ p+2q, \ldots p+(r-1)q,\ p+rq, \ldots$$

be successively substituted for x; then y becomes

$$ap+b,\ (ap+b)+aq,\ (ap+b)+2aq,$$
$$\ldots (ap+b)+(r-1)aq,\ (ap+b)+rq, \ldots;$$

and these expressions form an arithmetical progression whose first term is $ap+b$, and common difference aq.

Afternoon, 3 *to* 6.

GEOMETRY AND TRIGONOMETRY.

1. *Given two finite right lines, one divided in any manner into a number of segments, and the other undivided; show how to divide the latter similarly to the former.*

[Euclid, vi. 10.]

2. *Given two rectilinear figures, each of any magnitude and form; show how to construct a third, which shall be similar to one of them in form, and equal to the other of them in area.*

[Euclid, vi. 25.]

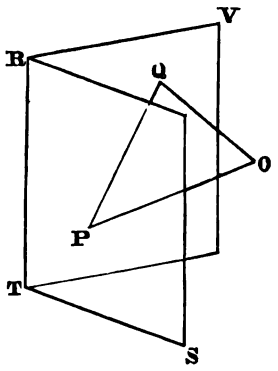
3. *Given a point and a plane, the point being supposed not to be in the plane; show how to draw through the point a right line which shall be perpendicular to the plane.*

[Euclid, xi. 11.]

4. *Two perpendiculars being supposed let fall from a common point upon two intersecting planes; show that, whatever be the position of the point, the plane of connection of the perpendiculars is perpendicular to the line of intersection of the planes.*

Let OP, OQ be the perpendiculars from any point O on the planes RS, TV; RT their intersection. Then [Euclid, xi. 18], ∵ OP is perpendicular to the plane RS, ∴ the plane OPQ which contains OP is perpendicular to the plane RS; and ∵ OQ is perpendicular to the plane TV, ∴ the plane OPQ is also perpendicular to the plane TV; hence the plane OPQ being perpendicular to the two planes RS, TV, it is perpendicular [Euclid, xi. 19] to their intersection RT.

Fig. 1.



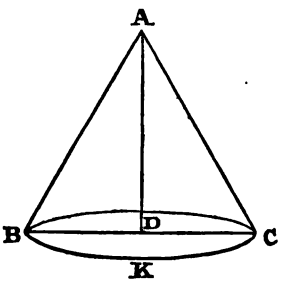
5. *An equilateral triangle being supposed to revolve round the line through its vertex perpendicular to its base; show that the area of the cone generated by either of its sides is double that of the circle generated by either half of its base.*

Let ABC be the equilateral triangle, AD perpendicular to the base BC, BKC the circular base swept out by BC, then the area of the cone = rectangle contained by the slant side AB and half the perimeter of the circular base BKC; and that of the circle BKC = the rectangle contained by its radius BD and half its perimeter; but

$$BD = \tfrac{1}{2}BC = \tfrac{1}{2}AB;$$

∴ the second area is one half the first, which is the result required.

Fig. 2.



6. *Three cylinders, of equal altitudes, being supposed to have for bases the three circles described on the three sides of*

a right-angled triangle as diameters; show that the volume of the greatest of them is equal to the sum of those of the remaining two.

Let a, b, c be the numbers of inches, feet, &c. in the sides of the triangle, c being opposite to the right angle; then $c^2=a^2+b^2$: also, let h be the number of inches, feet, &c. in the height of the cylinders; then the volumes of the cylinders in cubic inches, feet, &c. are

$$\frac{\pi a^2}{4}.h, \quad \frac{\pi b^2}{4}.h, \quad \frac{\pi c^2}{4}.h;$$

but
$$\frac{\pi a^2 h}{4}+\frac{\pi b^2 h}{4}=\frac{\pi}{4}(a^2+b^2)h=\frac{\pi c^2 h}{4},$$

which proves the theorem.

7. *A cone and hemisphere, being supposed to have a common base, and to lie at opposite sides of it; required the ratio of the altitude of the cone to the radius of the hemisphere, in order that the volumes of both solids should be equal.*

Let a be the radius of the common base, h the height of the cone; then the volume of the hemisphere is, by the formula, $\frac{2}{3}\pi a^3$; that of the cone is $\frac{1}{3}$ height $\times$ area of base $=\frac{1}{3}\pi a^2 h$; and these are equal;

$$\therefore \ \tfrac{2}{3}\pi a^3=\tfrac{1}{3}\pi a^2 h,$$
$$\therefore \ 2a=h,$$

or the ratio of the height to the radius is two to one.

8. *If x_1y_1, x_2y_2, x_3y_3 be the Cartesian coordinates of three points P_1, P_2, P_3 in a plane, with respect to any pair of rectangular axes in the plane; determine, to the same axes, the equations of the parallel, and of the perpendicular through P_3, to the right line P_1P_2.*

The line P_1P_2 is

$$\frac{y-y_1}{y_1-y_2}=\frac{x-x_1}{x_1-x_2},$$

or
$$y-y_1=\frac{y_1-y_2}{x_1-x_2}(x-x_1);$$

$\therefore$ the lines through P_3 parallel and perpendicular respectively to this line are

$$y-y_3=\frac{y_1-y_2}{x_1-x_2}(x-x_3)\ \{\text{parallel}\},$$

$$y-y_3=-\frac{x_1-x_2}{y_1-y_2}(x-x_3)\ \{\text{perpendicular}\}.$$

9. *The equation of a circle in rectangular coordinates being*

$$x^2+y^2+2r\cos\phi\,.\,x+2r\sin\phi\,.\,y+r^2=0\ ;$$

determine the coordinates of the centre, the square of its radius, and the common length of the two equal tangents from the origin.

Taking the equation in the form

$$x^2+2r\cos\phi\,.\,x+y^2+2r\sin\phi\,.\,y=-r^2,$$

and completing the squares on the left, we obtain

$$\begin{aligned}(x+r\cos\phi)^2+(y+r\sin\phi)^2&=r^2\cos^2\phi+r^2\sin\phi-r^2\\&=r^2-r^2=0.\end{aligned}$$

Comparing this with the general equation

$$(x-h)^2+(y-k)^2=c^2,$$

the coordinates of the centre are seen to be

$$-r\cos\phi,\quad -r\sin\phi\ ;$$

the radius is zero; and the two tangents are coincident, their length being that of the straight line from the origin to the centre, viz. $\sqrt{(r\cos\phi)^2+(r\sin\phi)^2}$ or r.

10. *Assuming the trigonometrical formulæ for* $\sin(A+B)$ *and for* $\cos(A+B)$ *in terms of the sines and cosines of* A *and* B*: deduce from them those for* $\tan(A+B+C)$ *and for* $\cot(A+B+C)$ *in terms of the tangents and cotangents of* A, B, C *respectively.*

The best method is as follows:—

$$\begin{aligned}\sin(A+B)&=\sin A\cos B+\cos A\sin B,\\ \cos(A+B)&=\cos A\cos B-\sin A\sin B\ ;\end{aligned}$$

$$\therefore \tan(A+B) = \frac{\sin A\cos B+\cos A\sin B}{\cos A\cos B-\sin A\sin B}$$

$$= \frac{\tan A+\tan B}{1-\tan A\tan B}.$$

Hence $$\tan(A+B+C) = \frac{\tan(A+B)+\tan C}{1-\tan(A+B)\tan C}$$

$$= \frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan B\tan C-\tan C\tan A-\tan A\tan B}.$$

Similarly

$$\cot(A+B+C) = \frac{\cot A\cot B\cot C-\cot A-\cot B-\cot C}{\cot B\cot C+\cot C\cot A+\cot A\cot B-1}.$$

11. *Given, of a plane triangle, any two sides a and b, and the contained angle* C*; express, in terms of them, the remaining side c, the two remaining angles* A *and* B*, and the area* Δ.

We have $$c^2 = a^2+b^2-2ab\cos C; \quad \ldots \quad \text{I.}$$

$$\left.\begin{aligned}\sin A &= \frac{a\sin C}{c},\\ \sin B &= \frac{b\sin C}{c};\end{aligned}\right\} \text{ and we have } c \text{ above.} \quad \ldots \quad \text{II.}$$

Or thus, without using the surd value for c,

$$\frac{\sin A}{\sin B} = \frac{\sin(B+C)}{\sin B} = \frac{a}{b}$$

$$= \frac{\sin B\cos C+\cos B\sin C}{\sin B} = \cot B\sin C+\cos C;$$

$$\therefore \quad \cot B = \left(\frac{a}{b}-\cos C\right)\div\sin C$$

$$\left.\begin{aligned}&= \frac{a-b\cos C}{b\sin C};\\ \text{similarly} \quad \cot A &= \frac{b-a\cos C}{a\sin C};\end{aligned}\right\} \quad \ldots \quad \text{III.}$$

and $$\Delta = \frac{ab}{2}\sin C. \quad \ldots \quad \text{IV.}$$

12. *The measured lengths of the three sides a, b, c of a plane triangle are* 13, 14, 15 *feet respectively: calculate, in square feet, its area* Δ, *and, in ordinary fractions, the sines of its three angles* A, B, C.

We have always

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)},$$
$$s = \frac{a+b+c}{2}.$$

In the given example

$$s = 21, \quad s-a=8, \quad s-b=7, \quad s-c=6;$$
$$\therefore \quad \Delta = \sqrt{21 \cdot 8 \cdot 7 \cdot 6}$$
$$= 84 \qquad \text{in square feet.}$$

Again,

$$\sin A = \frac{2\Delta}{bc} = \frac{168}{14 \cdot 15} = \frac{4}{5},$$
$$\sin B = \frac{2\Delta}{ca} = \frac{168}{13 \cdot 15} = \frac{56}{65},$$
$$\sin C = \frac{2\Delta}{ab} = \frac{168}{13 \cdot 14} = \frac{12}{13}.$$

Wednesday, July 17, 1878.

Examiners, { Prof. HENRICI, Ph.D., F.R.S.
Rev. Prof. TOWNSEND, M.A., F.R.S.

Morning, 10 *to* 1.

ARITHMETIC AND ALGEBRA.

1. *Verify, by actual multiplication or otherwise, the algebraic identity*

$$(y-z)^3+(z-x)^3+(x-y)^3=3(y-z)(z-x)(x-y).$$

The following identity should be remembered; it is of very frequent use:

$$a^3+b^3+c^3-3abc=(a+b+c)(a^2+b^2+c^2-bc-ca-ab).$$

Using this identity write for a, $y-z$; for b, $z-x$; and for c, $x-y$; we at once obtain

$$a+b+c=0;$$

and therefore

$$\begin{aligned} a^3+b^3+c^3-3abc &= (y-z)^3+(z-x)^3+(x-y)^3 \\ &\quad -3(y-z)(z-x)(x-y) \\ &= 0, \end{aligned}$$

which is equivalent to the result required.

2. *Find by actual multiplication or otherwise, and arrange in ascending powers of x, supposed to be less than unity, the square of the progression*

$$1+\tfrac{1}{2}x+\tfrac{1}{2}\cdot\tfrac{3}{4}x^2+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}x^3+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}\cdot\tfrac{7}{8}x^3+\&c. \text{ to infinity.}$$

This may be easily multiplied out as far as necessary; but by applying the Binomial Theorem we get a general result; for the given series is the expansion of $\frac{1}{\sqrt{1-x}}$, and therefore its square is

$$\frac{1}{1-x} \text{ or } 1+x+x^2+x^3+\ldots \text{ to infinity,}$$

the same result as will be obtained by ordinary squaring.

3. *Extract, by the ordinary process or otherwise, and arrange again in ascending powers of x supposed as before to be less than unity, the square root of the progression*

$$1+x+x^2+x^3+x^4+\ldots \&c. \text{ to infinity.}$$

This question is the converse of question 2; and the result is

$$\sqrt{\frac{1}{1-x}}=(1-x)^{-\frac{1}{2}}=1+\tfrac{1}{2}x+\tfrac{1}{2}\cdot\tfrac{3}{4}x^2+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}x^3+\ldots$$

We may notice that unless x is less than unity both series are divergent, and our processes and results become meaningless.

4. *Given that, in an Arithmetical Progression of n terms commencing with unity, the sum is equal to the square of the*

number of terms; determine, by any method, the common difference.

Taking the simplest method,

$$\begin{aligned} &\text{1st term} = 1, \\ \text{1st term} + &\text{2nd term} = 4, \\ \therefore \quad &\text{2nd term} = 4-1 = 3, \\ \text{and} \quad &\text{1st term} = \quad 1, \\ \therefore \ &\text{common difference is 2.} \end{aligned}$$

But this method does not prove that such a series can exist, to which objection the following method is not open. Let d be the common difference, then the sum of n terms is

$$\frac{n}{2}(2+\overline{n-1}|d) \text{ or } n^2 . \frac{d}{2} + n . \frac{2-d}{2};$$

this is to be always equal to n^2, and we see that this will require d to be equal to 2, the same result as before.

It is worth noticing that the value $d=2$ secures the fulfilment of two conditions—viz. that in the sum the coefficient of n^2 is 1, and of n, 0.

5. *Given that, in a Geometrical Progression of n terms commencing with unity, the sum is a number consisting of n digits equal to unity; determine, by any method, the common ratio.*

This question is very similar to the preceding.

First method:

$$\begin{aligned} &\text{1st term} = 1, \\ &\text{Sum of first 2 terms} = 11, \\ \therefore \quad &\text{2nd term} = 10, \\ \therefore \ &\text{common ratio} = 10. \end{aligned}$$

Second method:

$$\text{The sum to } n \text{ terms} = \frac{r^n-1}{r-1} = 111\ldots n \text{ digits} = \frac{10^n-1}{10-1};$$

hence $r=10$.

The series therefore is 1, 10, 100, $\therefore$ from which the truth of the given theorem is at once apparent.

6. *The sum of two numbers is* 1878, *and their product is* 880821; *determine them completely, each to the last integer figure.*

Let x, y be the numbers, then

$$x+y=1878,$$
$$xy=880821;$$
$$\therefore (x-y)^2=(1878)^2-4.880821$$
$$=4.9.(313^2-97869)$$
$$=3600;$$
$$\therefore x-y=60.$$

We need not take both signs here; and

$$\left.\begin{aligned} x&=969\\ y&=909 \end{aligned}\right\} \text{exactly.}$$

7. *Calculate, to the nearest halfpenny, the true present value of a bill for* £152 8*s.*, *payable on December* 31, *and discounted on July* 17 *at* $3\frac{1}{2}$ *per cent. interest per annum.*

Number of days the bill has to run is 167, and the amount of £100 in this time is £$(100+\frac{167}{365}$ of $3\frac{1}{2})$ or £101·6014, retaining 7 significant figures as we wish the result true to about ·001£, or about 6 significant figures in our result.

Now since the present value of £101·6014 is £100, the present value of £152 8*s.* under the same conditions is £$\frac{152\cdot4\times100}{101\cdot6014}$ = £149·9980 . . which falls short of £150 by about ·002£, or about one halfpenny: hence the present value to the nearest halfpenny is £149 19*s.* $11\frac{1}{2}d.$

8. *The operatives in a factory being supposed to consist of a men, b women, c boys, and d girls; required the entire number of different combinations of p men, q women, r boys, and s girls, that can be told off from among them to any particular work.*

The number of combinations of p men that can be selected

is $\frac{|a}{|p\,|a-p}$, of q women $\frac{|b}{|q\,|b-q}$, of r boys $\frac{|c}{|r\,|c-r}$, of s girls $\frac{|d}{|s\,|d-s}$.

But each set of men can be combined with each set of women, with each set of boys, and with each set of girls; and hence the number of combinations of p men, q women, r boys, and s girls that can be told off to the work is obtained by multiplying the numbers above, the product being

$$\frac{|a\,|b\,|c\,|d}{|p\,|q\,|r\,|s\,|a-p\,|b-q\,|c-r\,|d-s}.$$

9. *Find the values of x, y, z which satisfy the three algebraical equations:—*

$$ax+by+cz=p^2, \quad \ldots\ldots \quad (1)$$
$$fx+gy+hz=q^2, \quad \ldots\ldots \quad (2)$$
$$x^2+y^2+z^2=r^2. \quad \ldots\ldots \quad (3)$$

For solution of equations of this form when $p=0$, $q=0$, see Todhunter's Algebra, § 385. To reduce the above to this form substitute

$$x_1+x_2 \text{ for } x,$$
$$y_1+y_2 \text{ for } y,$$
$$z_1+z_2 \text{ for } z;$$

then
$$ax_1+by_1+cz_1+ax_2+by_2+cz_2=p^2,$$
$$fx_1+gy_1+hz_1+fx_2+gy_2+hz_2=q^2.$$

Now if we choose for x_2, y_2, z_2 any of an indefinite number of values satisfying the *two* equations

$$ax_2+by_2+cz_2=p^2, \quad \ldots\ldots \quad (4)$$
$$fx_2+gy_2+hz_2=q^2, \quad \ldots\ldots \quad (5)$$

x_1, y_1, z_1 must satisfy

$$ax_1+by_1+cz_1=0, \quad \ldots\ldots \quad (6)$$
$$fx_1+gy_1+hz_1=0, \quad \ldots\ldots \quad (7)$$

and must therefore have the values

$$k(bh-cg), \quad k(cf-ah), \quad k(ag-bf).$$

Substituting in equation (3),

$$\begin{aligned} &k^2\{(bh-cg)^2+(cf-ah)^2+(ag-bf)^2\} \\ &\quad+2k\{(bh-cg)x_2+(cf-ah)y_2+(ag-bf)z_2\} \\ &\quad+x_2^2+y_2^2+z_2^2=r^2 \end{aligned} \quad \ldots\ldots\ldots\ldots \quad (8)$$

Now we can make x_2, y_2, z_2 satisfy the *three* equations

$$\left.\begin{aligned} &(bh-cg)x_2+(cf-ah)y_2+(ag-bf)z_2=0, \\ &ax_2+by_2+cz_2=p^2, \\ &fx_2+gy_2+hz_2=q^2; \end{aligned}\right\} \quad (9)$$

and then we have, since the middle term of (8) vanishes

$$k^2=\frac{r^2-(x_2^2+y_2^2+z_2^2)}{(bh-cg)^2+(cf-ah)^2+(ag-bf)^2},$$

which enables us to find x_1, y_1, z_1 since x_2, y_2, z_2 are already given by the solution of equation (9).

10. *Given that the two quadratic functions*

$$a_1x^2+2h_1x+b_1, \text{ and } a_2x^2+2h_2x+b_2,$$

have a common factor; prove, by any method, the equation of condition

$$(a_2b_1+a_1b_2-2h_1h_2)^2=4(h_1^2-a_1b_1)(h_2^2-a_2b_2).$$

Let the factors of the first expression be $x+p, x+q$; and of the second, $x+p, x+r$; $x+p$ being the common factor. Then, by the theory of quadratic expressions,

$$p+q=\frac{2h_1}{a_1}, \quad p+r=\frac{2h_2}{a_2},$$

$$pq=\frac{b_1}{a_1}, \qquad pr=\frac{b_2}{a_2}.$$

$$\begin{aligned} \text{Now } (p-q)(p-r)&=(p+q)(p+r)-2pq-2pr \\ &=\frac{4h_1h_2}{a_1a_2}-\frac{2b_1}{a_1}-\frac{2b_2}{a_2} \\ &=\frac{4h_1h_2-2b_1a_2-2b_2a_1}{a_1a_2}; \end{aligned}$$

but $$(p-q)^2=(p+q)^2-4pq=\frac{4h_1^2}{a_1^2}-\frac{4b_1}{a}$$

$$=\frac{4}{a_1^2}(h_1^2-a_1b_1).$$

Similarly

$$(p-r)^2=\frac{4}{a_2^2}(h_2^2-a_2b_2).$$

Hence

$$\frac{16}{a_1^2a_2^2}(h_1^2-a_1b_1)(h_2^2-a_2b_2)=(p-q)^2(p-r)^2$$

$$=\left(\frac{4h_1h_2-2b_1a_2-2a_1b_2}{a_1a_2}\right)^2,$$

or

$$4(h_1^2-a_1b_1)(h_2^2-a_2b_2)=(2h_1h_2-b_1a_2-b_2a_1)^2,$$

which is the *necessary* condition.

It is also sufficient; for assuming this identity as given, and supposing the factors of the expressions to be $x+p_1$ and $x+q_1$, $x+p_2$ and $x+q_2$ respectively, we have

$$p_1+q_1=-\frac{2h_1}{a_1}, \quad p_2+q_2=-\frac{2h_2}{a_2},$$

$$p_1q_1=\frac{b_1}{a_1}, \qquad p_2q_2=\frac{b_2}{a_2};$$

and substituting these values in the identity, we find it reduces to

$$[(p_1+q_1)^2-4p_1q_1][(p_2+q_2)^2-4p_2q_2]$$
$$=[(p_1+q_1)(p_2+q_2)-2p_1q_1-2p_2q_2]^2$$

when cleared of fractions; or

$$(p_1-q_1)^2(p_2-q_2)^2$$
$$=[(p_1+q_1)(p_2+q_2)-2p_1q_1-2p_2q_2]^2;$$

$$\therefore \quad (p_1-q_1)(p_2-q_2)$$
$$=\pm(p_1p_2+q_1q_2+q_1p_2+p_1q_2-2p_1q_1-2p_2q_2).$$

With the upper sign

$$(p_1-p_2)(q_1-q_2)=0,$$

with the lower sign

$$(p_1-q_2)(p_2-q_1)=0\,;$$

$\therefore$ either $p_1=p_2$, $q_1=q_2$, $p_1=q_2$, or $p_2=q_1$;

or in other words, there is, if this condition is satisfied, a common factor of the two given expressions.

11. *Assuming the fundamental properties of logarithms, state, and prove, the ordinary rules by which the logarithm of a number not given in the tables, and conversely the number corresponding to a logarithm not given in the tables, may be calculated from the tables.*

[Todhunter's Trigonometry, §§ 176, 177, &c.]

12. *Given that, to ten decimal places, in the ordinary system,* $\log 2=0{\cdot}3010299561$, $\log 3=0{\cdot}4771212546$, *calculate to the same number of places, the values of* $\log 4{\cdot}5$, *of* $\log 6{\cdot}75$, *and of* $\log 10{\cdot}125$ *in the same system.*

Since $4{\cdot}5=\frac{9}{2}=\frac{3^2}{2}$,

$$\therefore \log 4{\cdot}5=2\log 3-\log 2$$
$$=\cdot 9542425092-\cdot 3010299561$$
$$=\cdot 6532125531.$$

Again, $6{\cdot}75=\frac{27}{4}=\frac{3^3}{2^2}$,

$$\therefore \log 6{\cdot}75=3\log 3-2\log 2$$
$$=\cdot 8293038516.$$

Again, $10{\cdot}125=\frac{81}{8}=\frac{3^4}{2^3}$,

$$\therefore \log 10{\cdot}125=4\log 3-3\log 2$$
$$=1{\cdot}0053951501.$$

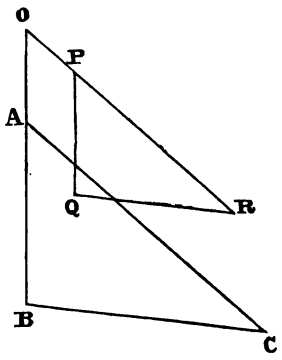
Afternoon, 3 to 6.

GEOMETRY AND TRIGONOMETRY.

1. *Prove that two triangles are similar if the sides of one are respectively parallel to those of the other.*

Various cases may arise, as the relative positions of the triangles may vary, but the general principle will be the same in the proof of all. We shall therefore take but one case:—Let ABC, PQR be two triangles whose sides are parallel, each to each, viz. AB to PQ, AC to PR, BC to QR; and suppose PR, AB to meet when produced in O.

Fig. 3.



Then

$\because$ PQ is parallel to AB,

$\therefore$ $\angle RPQ = \angle ROB$;

and

$\because$ PR is parallel to AC,

$\therefore$ $\angle ROB = \angle CAB$,

$\therefore$ $\angle CAB = \angle RPQ$.

Similarly the other angles are equal to each other; and therefore the triangles are similar.

A very simple method of proving the whole of the cases that may arise depends on the fact that if one pair of straight lines be parallel to another pair, the angles they make are equal or supplementary.

Hence the three angles of the triangle PQR are equal or supplementary to those of ABC. Denoting each angle by a single letter, we have

$$A = P \text{ or } \pi - P,$$
$$B = Q \text{ or } \pi - Q,$$
$$C = R \text{ or } \pi - R.$$

The number of cases seems at first 8; but they practically reduce to 4, viz.:—

(1)	(2)	(3)	(4)
$A=P,$	$A=P,$	$A=P,$	$A=\pi-P,$
$B=Q,$	$B=Q,$	$B=\pi-Q,$	$B=\pi-Q,$
$C=R,$	$C=\pi-R,$	$C=\pi-R,$	$C=\pi-R.$

We shall show that the third and fourth are impossible, and that the second, when possible, reduces to the first, which is, in fact, the result we wish to establish.

Adding the three equations in each case

$$A+B+C=P+Q+\pi-R, \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\text{or} \quad 2\pi+P-Q-R, \quad . \quad . \quad . \quad . \quad (3)$$

$$\text{or} \quad 3\pi-P-Q-R; \quad . \quad . \quad . \quad . \quad (4)$$

but $A+B+C=P+Q+R=\pi$ always.

Hence we have, by reducing,

$$P+Q=R, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$\pi+P=Q+R, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$2\pi \quad =0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

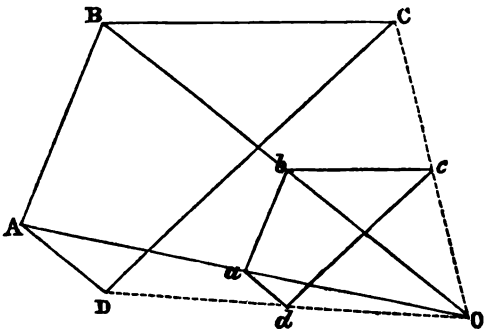
Now (3) and (4) are clearly absurd; and (2) is only true when $R=\frac{\pi}{2}$, or $\pi-R=R$; so that in this case $\left.\begin{matrix}A=P,\\B=Q,\\C=R,\end{matrix}\right\}$ and hence the triangles are always similar.

2. *If two similar rectilinear figures be situated in such a manner that two sides of the one are parallel respectively to the two homologous sides of the other, prove that every side of the one is parallel to the homologous side of the other, and that the lines joining homologous vertices all pass through a common point.*

Let ABCD, *abcd* be two similar rectilinear figures such that the sides AB, BC of the first are parallel to the homo-

logous sides ab, bc of the second; then shall every side of the first be parallel to the homologous side of the second,

Fig. 4.



and the lines Aa, Bb, Cc, Dd shall all pass through a common point.

It is clear that ab, bc may fall in one of two ways—either so that A and a are towards the same parts in AB, ab, or not; in the first case Aa, Bb do not meet unless produced. Let us first consider this case (fig. 4), and suppose Aa, Bb, Cc, Dd joined.

Then

$\because$ BC is parallel to bc, and Cc meets them,

$\therefore$ $\angle BCc + \angle Ccb = 2$ right angles,

$= \angle BCD + \angle DCc + \angle Ccb$;

but $\angle BCD = \angle bcd$,

and they are respectively on the same sides of BC, bc.

Hence

$$\angle BCD + \angle DCc + \angle Ccb$$
$$= \angle DCc + \angle Ccb + \angle bcd$$
$$= \angle DCc + \angle Ccd;$$

$\therefore$ $\angle DCc + \angle CcD = 2$ right angles;

$\therefore$ CD and cd are parallel;

similarly DA and da are parallel.

Hence pairs of homologous sides are parallel. A similar proof applies, however many sides there may be in the figures. It is also clear that not only A, a, but also B, b, C, c, D, d all lie towards the same parts of the sides containing them.

Suppose Aa and Bb meet in O;

then $\because$ AB is parallel to ab;

$$\therefore \quad AO : aO :: BO : bO :: AB : ab.$$

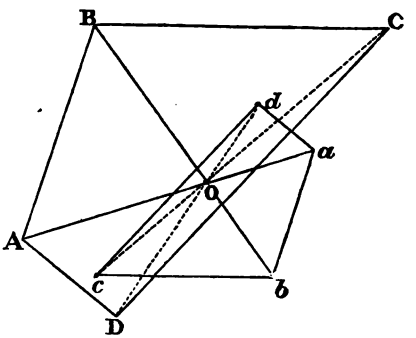
Hence Bb divides Aa, and is itself divided at O, in the ratio of any pair of homologous sides.

Hence Cc divides Aa in the same ratio, and therefore Cc passes through O; similarly Dd passes through O.

Thus we have proved the second part of our proposition.

In the second case (fig. 5) the same method may be applied; only here Aa, Bb, Cc, Dd cut each other in O without being produced.

Fig. 5.



Thus

$\because$ BC is parallel to bc,

$\therefore$ alternate angles are equal, viz. $\angle$ BCc and $\angle$ Ccb;

but $\angle BCD = \angle bcd$;

hence the remainders cCD, Ccd are equal; and these are alternate angles;

$$\therefore \quad CD \text{ is } \| \; cd,$$

and so on.

NOTE 1. The point O is called the centre of similitude of the two figures, direct in the first case, inverse in the second.

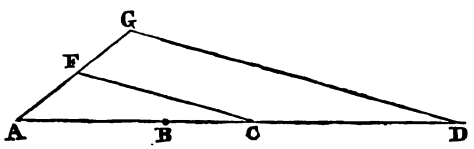
NOTE 2. If we are only given one side parallel to one side, it does not follow that the other sides are parallel.

3. *On a straight line three points* A, B, C *are given,* B *lying between* A *and* C. *Find a point,* D, *which divides the line* AC *externally in the same ratio as* B *does internally.*

[See Euclid, vi. 10, of which this problem is an extension.]

From A draw any line AG, making an angle with AB; of the parts AB, BC, let AB be the greater; make

Fig. 6.



AG = AB, and cut off from GA, GF = BC, and from G draw GD parallel to FC, meeting AB in D, the point required.

Then

$\because$ DG is parallel to FC,

$\therefore$ AD : CD :: AG : FG

:: AB : BC;

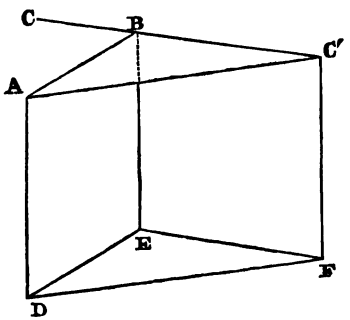
$\therefore$ AC is divided externally by D in the ratio in which it is divided internally by B.

4. *Two angles in space have the limits of one parallel respectively to those of the other. Prove that the angles are equal or supplemental, and that their planes are parallel.* [See Euclid, xi. 10 & 15.]

Let AB, BC be parallel respectively to DE, EF. Now, if AB lie in the same direction as DE (that is to say, if AD, BE join AB, DE towards the same parts: see Euclid,

i. 33), and BC in the same direction as EF, then the first part of the proof proceeds as in Euclid xi. 10; if, however, one (or both) the lines AB, BC lie in the opposite direction to DE, EF, produce it (or them) backwards to B′ or C′ (or both) as may be necessary, and then proceed as in the proposition. For instance, suppose AB is in the same direction as DE, but BC is in the opposite direction to EF; produce BC backwards to C′, making BC′=EF; also make BA=ED; join AD, BE, C′F, C′A, DF.

Fig. 7.



∵ BC′ is equal and parallel to EF,

∴ BE and C′F, which join their extremities towards the same parts, are equal and parallel.

Similarly AD and BE are equal and parallel.

Hence AD and C′F are equal and parallel; and hence AC′ and DF are also equal and parallel.

Therefore, by Euclid, i. 8, the triangles ABC′, DEF are equal in all respects, and

$$\angle ABC' = \angle DEF;$$

but $\angle ABC'$ is the supplement of $\angle ABC$;

$\therefore$ $\angle ABC$ is the supplement of DEF.

The second part is equivalent to Euclid, xi. 15.

5. *Show that every plane cuts a sphere in a circle. Determine the radius of the circle in which a plane cuts a*

sphere of radius 13 *inches, if the distance of the plane from the centre of the sphere be* 12 *inches.*

For the first part see Wilson's Solid Geometry, p. 52.

For the second part, let O be the centre of the sphere, OP the perpendicular on the plane, Q any point on the curve of section; then P is the centre of this curve, and PQ is a radius.

$\because$ OPQ is a right angle,

$\therefore$ $OQ^2=OP^2+PQ^2$,

or $13^2=12^2+PQ^2$;

$\therefore$ $PQ=5$,

and area of the circle of section

$=\pi . PQ^2=25\pi$,

all being expressed in inches.

6. *The altitude of a right circular cone equals the circumference of its base. Calculate the volume, and the area of the whole surface of the cone, the radius of the base being given.*

Let a=radius of base,

l=slant side of the cone,

h=height of the cone;

then $h=2\pi a$=perimeter of base,

$l^2=h^2+a^2=a^2(4\pi^2+1)$.

Volume $=\frac{1}{3} . h$ (area of base)

$=\frac{1}{3} . \pi a^2 h=\frac{2}{3}\pi^2 a^3$.

Again,

curved area $=\frac{1}{2} . l$ (perimeter of base)

$=\pi a l=\pi a^2\sqrt{4\pi^2+1}$;

and flat area, *i. e.* of base, $=\pi a^2$;

$\therefore$ whole area $=\pi a^2\{\sqrt{4\pi^2+1}+1\}$.

7. *Determine the sine and cosine of an angle whose tangent equals* -2, *and whose sine is positive.*

Here $\tan\Theta=-2$, required $\sin\Theta$ and $\cos\Theta$,

$$\sin\theta=\frac{\tan\theta}{\sqrt{1+\tan^2\theta}}=\frac{2}{\sqrt{5}}.$$

We take the positive sign for the root, as the sine is given positive. Again

$$\tan\theta=-2=\frac{\sin\theta}{\cos\theta}=\frac{2}{\sqrt{5}\cos\theta};$$

$$\therefore\quad \cos\theta=\frac{-1}{\sqrt{5}}.$$

8. A, B, C *being the angles in a triangle, show that*

$$\tan A\tan B\tan C=\tan A+\tan B+\tan C.$$

$$\tan A=\tan(180°-B-C)=-\tan(B+C)$$
$$=-\frac{\tan B+\tan C}{1-\tan B\tan C};$$

whence the result immediately follows.

9. *A vertical pole (more than* 100 *feet high) consists of two parts, the lower being* $\frac{1}{3}$ *of the whole. From a point in a horizontal plane through the foot of the pole, and* 40 *feet from it, the upper part subtends an angle whose tangent is equal to* $\frac{1}{2}$. *Determine the height of the pole.*

Let AB, BC be the two parts of the pole, P the point of observation; let

$$AB=x,\ BC=2x,\ AC=3x.$$

Then we have

$$\tan BPC=\tfrac{1}{2};$$

but $\quad \angle BPC=\angle CPA-\angle BPA;$

$$\therefore\quad \tan BPC=\tan(CPA-BPA)$$
$$=\frac{\tan CPA-\tan BPA}{1+\tan CPA\tan BPA}$$
$$=\frac{\frac{3x}{40}-\frac{x}{40}}{1+\frac{3x^2}{1600}}=\tfrac{1}{2};$$

$$\therefore\quad 3x^2+1600=160x.$$

Hence $x=\frac{40}{3}$ or 40; but $3x>100$.

Hence $x=40$, $3x=AC=120$ feet.

10. *Through the point of intersection of the straight lines whose equations are*

$$2x-3y+7=0,$$
$$x+4y+3=0,$$

a straight line is drawn at right angles to the axis of x, and another at right angles to the line whose equation is

$$x+y+1=0.$$

Find the equations of these lines.

These equations may be obtained by the ordinary process of first finding the point of intersection of the given lines, and then writing down the equations required; or it may be more simply solved thus:—

The equation

$$\lambda(2x-3y+7)+\mu(x+4y+3)=0$$

represents a straight line, since it is of the first degree. It also passes through the point common to the given pair of lines; and by suitably choosing the numerical quantities λ and μ we can make it represent any line through this point.

The equation rearranged is

$$x(2\lambda+\mu)+y(-3\lambda+4\mu)+(7\lambda+3\mu)=0.$$

Now, if this line is to be perpendicular to the axis of x, the coefficient of y must vanish, *i. e.* $3\lambda=4\mu$; hence $\frac{\lambda}{4}=\frac{\mu}{3}$.

Substituting these ratios the equation is

$$11x+37=0.$$

Again, in the second case the line sought must be perpendicular to $x+y+1=0$;

hence $$(2\lambda+\mu)+(-3\lambda+4\mu)=0,$$

and $$\frac{\lambda}{5}=\mu.$$

In this case the equation becomes

$$11x-11y+38=0.$$

11. *Find the equations of the tangents to the circle*

$$x^2+y^2+2Ax+2By+C=0$$

which are parallel to the line

$$x+2y-6=0.$$

The equation to the circle may be thrown into the form

$$(x+A)^2+(y+B)^2=A^2+B^2-C=R^2.$$

Now any point on this circle may be represented by the coordinates

$$x=-A+R\cos\phi,$$
$$y=-B+R\sin\phi,$$

where ϕ is the angle that the radius from (x, y) to the centre of the circle makes with the axis of x.

Also the tangent at (x, y) may be written

$$(X+A)(x+A)+(Y+B)(y+B)=R^2,$$

where X, Y are current coordinates.

This is the same as

$$(X+A)\cos\phi+(Y+B)\sin\phi=R.$$

But this tangent is to be parallel to

$$X+2Y-6=0;$$

$$\therefore \quad \tan\phi=2,$$

$$\cos\phi=\pm\frac{1}{\sqrt{5}},$$

$$\sin\phi=\pm\frac{2}{\sqrt{5}},$$

and the tangents are

$$(X+A)+2(Y+B)=\pm R\sqrt{5}$$
$$=\pm\sqrt{5(A^2+B^2-C)},$$

or $$X+2Y+A+2B\pm\sqrt{5(A^2+B^2-C)}=0.$$

The question can also be solved by assuming for the equation of a tangent $x+2y=d$, and then making d such that the points of intersection of this line and the circle coincide.

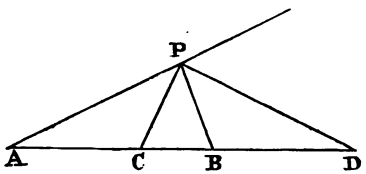
12. *Prove by elementary or coordinate geometry that the locus of points whose distances from two fixed points have a constant ratio, is a circle.*

Both proofs are worthy of notice, the first for its elegance, the second for its simplicity.

PROOF BY PURE GEOMETRY.

Let A, B be the given points, P a point on the locus required, such that PA : PB is in a given ratio.

Fig. 8.



Divide AB internally in C, and externally in D, in this ratio. Then C and D are fixed points. Join PC, PD.

Then

$\because$ PA : PB :: AC : CB (constr.),

$\therefore$ PC bisects the angle APB (Euc. vi. 3).

Again

$\because$ PA : PB :: AD : BD,

$\therefore$ PD bisects the exterior angle at B (Euc. vi. A).

Hence PC, PD bisecting respectively two adjacent angles are at right angles.

Hence CPD is a right angle; hence P is on the circle whose diameter is CD.

PROOF BY COORDINATE GEOMETRY.

Let the axis of y bisect AB, the axis of x, at right angles and let the coordinates of A, B, P be $-a$, 0; a, 0; and x, y

Then
$$PA^2=(x+a)^2+y^2,$$
$$PB^2=(x-a)^2+y^2;$$

but PA : PB in a constant ratio, say $m : 1$;

$$\therefore \quad PA^2 = m^2 PB^2;$$

$$\therefore \quad (x+a)^2 + y^2 = m^2\{(x-a)^2 + y^2\};$$

$$\therefore \quad x^2(1-m^2) + y^2(1-m^2) + 2ax(1+m^2) + a^2(1-m^2) = 0.$$

This represents a circle (since the coefficients of x^2 and y^2 are equal, and there is no xy term), except when the ratio is one of equality; and then we obtain a straight line bisecting AB at right angles, viz. $x=0$.

Wednesday, July 23, 1879.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
Rev. Prof. TOWNSEND, M.A., F.R.S.

Morning, 10 *to* 1.

ARITHMETIC AND ALGEBRA.

1. *Multiply together* $1\cdot\dot{3}\dot{4}$ *and* $2\cdot5\dot{6}\dot{7}$, *and extract the square root of the result correctly to seven significant figures.*

$$1\cdot\dot{3}\dot{4} = \frac{134-13}{90} = \frac{121}{90},$$

$$2\cdot5\dot{6}\dot{7} = \frac{2567-25}{990} = \frac{2542}{990};$$

$$\therefore \text{ product} = \frac{11\times2542}{8100} = \frac{27962}{8100} = 3\cdot45\dot{2}09876543\dot{.}$$

Next for the square root, we work to 4 figures and then divide for the remaining three by contracted division, thus [Todhunter's Algebra, § 246]:—

```
        3·452098 . . . (1·857983 . . .
        1
      ————————
28      245
        224
       ———
365     2120
        1825
       ———
3707     29598
         25949
        ———
3714     3649
         3342
        ———
          307
          297
         ———
           10
           11
          ——
           ..
```

Result 1·857983.

2. *At what rate of compound interest will a given sum be increased* 11*-fold in* 100 *years?* [log 11 = 1·0413927; log 1·1266 = 0·0517697; log 1·1267 = 0·0518083.]

Let x be the rate per cent. required, then £1 will, in 100 years, amount to $£\left(1+\frac{x}{100}\right)^{100} = £11$ by the question.

Hence

$$100 \log\left(1+\frac{x}{100}\right) = \log 11 = 1{\cdot}0413927;$$

$$\therefore \log\left(1+\frac{x}{100}\right) = {\cdot}0104139.$$

Now to use the given logarithms we require a small device: add log 11 to both sides; then

$$\log 11+\log\left(1+\frac{x}{100}\right)=1\cdot0518066;$$

$$\therefore\ \log\left(11+\frac{11x}{100}\right)=1\cdot0518066;$$

and the mantissa of this lies between those of the two given logs.

In the usual way,

difference for $1=386$,
difference found (in defect) $=17$;

$\therefore$ quantity to be subtracted is $\frac{17}{386}=\cdot04$.

Hence the significant figures of the number whose logarithm is 1·0518066 are 1126696; and since the characteristic is 1,

$$11+\frac{11x}{100}=11\cdot26696,$$

or
$$x=\frac{26\cdot696}{11}$$
$$=2\cdot427$$
$$=\text{rate per cent. required.}$$

3. *A man of 25 years of age can insure his life for £1000 by paying an annual premium of £18. Taking interest at 5 per cent., what will be an equitable composition for an annual subscription of £3?*

Let x be the number of payments on which the premium is calculated; then £1000 = amount of an unpaid annuity of £18 for x years at 5 per cent.

$$\therefore\quad 1000=\frac{18\times\{(1\cdot05)^x-1\}}{1\cdot05-1};$$

and we wish to find the present value of an annuity of £3 for the same time; this is

$$\frac{£3\times\{(1\cdot05)^x-1\}}{(1\cdot05)^{x-1}(1\cdot05-1)}.$$

Now

$$(1{\cdot}05)^x = 1 + \tfrac{50}{18} = \tfrac{34}{9},$$

from the first equation; and we require

$$\frac{3\times\{(1{\cdot}05)^x - 1\}}{(1{\cdot}05)^x\left(1-\frac{1}{1{\cdot}05}\right)}$$

$$= \frac{3\times 1{\cdot}05\times\frac{50}{18}}{\frac{34}{9}\times{\cdot}05}$$

$$= \frac{3\times 21\times 25}{34} = \frac{1575}{34}$$

$$= 46\tfrac{11}{34};$$

$\therefore$ composition required is

£46 6*s.* 5$\frac{3}{4}$*d.* nearly.

We may notice that we tacitly assume x to be an integer, which is really not the case; but, since x is eliminated, this does not invalidate our final result. In fact it is quite justifiable to suppose the formula $\frac{(1{\cdot}05)^x - 1}{1{\cdot}05 - 1}$, which increases with x, and gives the amount of an annuity of £1 for all integral values of x, to represent the amount for fractional values of x also.

4. *Ten English labourers can do as much excavating in six days as nine French labourers in seven days; a Frenchman receives one franc per cubic metre: how many pence must an Englishman receive per cubic yard that his daily earnings may be 5 per cent. more than a Frenchman's?*

[*A metre may be taken as equal to* 39$\frac{3}{8}$ *inches, and a franc to tenpence.*]

60 days' work of an English labourer
= 63 days' work of a French labourer.

$\therefore$ the Englishman does 5 per cent. more daily work than the Frenchman; and $\therefore$ if he receive 5 per cent. more wages, he will be paid at just the same rate per metre, *i. e.* one franc.

$=10d.$ for $\left(\frac{315}{8}\right)^3$ cubic inches,

$=\frac{10\times 8^3\times 36^3}{(315)^3}d.$ for a cubic yard,

$=\frac{10\times 8^3\times 4^3}{35^3}d.$ for a cubic yard,

$=7\frac{1}{2}d.$ nearly.

5. *The sum of the squares of two numbers is* 1105, *and their product is* 552 *times their difference; what are they?*

Let x, y be the numbers, then

$$x^2+y^2=1105,$$
$$xy=552(x-y).$$

Evidently we must proceed to form a quadratic in $x-y$. Subtract twice the second line from the first; then

$$(x-y)^2+1104(x-y)-1105=0,$$
$$i.e.\ [(x-y)-1][(x-y)+1105]=0;$$
$$\therefore\ x-y=1,\text{ or } -1105.$$

Take the first value, we get

$$x^2+y^2=1105,$$
$$xy=552;$$
$$\therefore\ (x+y)^2=2209=47^2;$$
$$\therefore\ x+y=\pm 47$$
$$\text{and}\quad x-y=1\ \Big\};$$
$$\therefore\ x=24,\ -23,$$
$$y=23,\ -24.$$

Again, taking the second value of x y, we get

$$x^2+y^2=1105,$$
$$xy=-552\times 1105;$$
$$\therefore\ (x+y)^2=-1105\times 1103;$$
$$\therefore\ x+y=\sqrt{-1103\times 1105};$$

hence x and y, which are both imaginary.

6. *There are m white men, and n black men; n being greater than m. Find the number of ways in which each white man may have one black servant. If a white man may have any number of servants, in how many ways may every black man have a master?*

First, the number of servants that may be chosen, one for each master, is ${}_nC_m = \frac{\lfloor n}{\lfloor m \lfloor n-m}$; and these may be permuted among the m masters in $\lfloor m$ ways; $\therefore$ result is $\frac{\lfloor n}{\lfloor n-m}$.

Or thus; the number of permutations of the n black men m at a time = the number of ways in which each white man can have one black servant; for m black men may be drafted off to the masters in specified order, in this number of ways.

Hence result $= {}_nP_m = \frac{\lfloor n}{\lfloor n-m}$.

Again, in the second part, any of the n black men may have one of the m masters; this gives n^m ways, which of course include the cases in which *not* every white man has a servant: *e.g.* one white man may have all the servants.

7. *Solve the equation*

$$\frac{x-a}{b}+\frac{x-b}{a}=\frac{b}{x-a}+\frac{a}{x-b}.$$

$$\frac{x-a}{b}+\frac{x-b}{a}=\frac{x(a+b)-(a^2+b^2)}{ab},$$

$$\frac{b}{x-a}+\frac{a}{x-b}=\frac{x(a+b)-(a^2+b^2)}{(x-a)(x-b)};$$

$$\therefore \quad \frac{x(a+b)-(a^2+b^2)}{ab}=\frac{x(a+b)-(a^2+b^2)}{(x-a)(x-b)};$$

$$\therefore\ x(a+b)-(a^2+b^2)=0,$$

and $$x = \frac{a^2+b^2}{a+b}.$$

Or $$ab=(x-a)(x-b)$$
$$=x^2-x(a+b)+ab;$$
$$\therefore \quad x\{x-(a+b)\}=0;$$
$$\therefore \quad x=0, \text{ or } a+b.$$

We have therefore three roots of the given equation, viz. 0, $a+b$, $\frac{a^2+b^2}{a+b}$.

8. *Under what conditions will* x^3+ax^2+bx+c *be divisible by* x^2+px+q?

The best way is to suppose the division performed; then the first term of the quotient will be x, and the other $\frac{c}{q}$; then

$$x^3+ax^2+bx+c$$
$$=(x^2+px+q)\left(x+\frac{c}{q}\right) \text{ identically.}$$

$$\therefore a=p+\frac{c}{q}; \quad \therefore aq=pq+c;$$

$$b=q+\frac{cp}{q}; \qquad bq=q^2+cp.$$

This is as simple a form of the conditions as it is possible to obtain.

9. *Sum the series*

$$1+2x+3x^2+4x^3+\ldots+nx^{n-1}.$$

Let the sum be called s; then

$$s=1+2x+3x^2+4x^3+\ldots+nx^{n-1},$$
$$s.x= \quad x+2x^2+3x^3+\ldots+\overline{n-1}\,x^{n-1}+nx^n;$$
$$\therefore \quad s(1-x)=1+x+x^2+\ldots+x^{n-1}-nx^n.$$

Sum the geometrical progression, and we obtain

$$s(1-x)=\frac{1-x^n}{1-x}-nx^n;$$

$$\therefore\ s=\frac{1-(1+n)x^n+nx^{n+1}}{(1-x)^2}.$$

10. *A sum of* £1000 *bearing interest at* 5 *per cent. is to be paid off in three annual instalments, the payments, including the interest due, to be the same each year, and the first payment to be due at the end of the first year. What must the yearly payment be?*

Let x=number of pounds required; then the debt in pounds at the end of the first year will be $(1000\times\frac{21}{20}-x)$, at the end of the second it will be $(1000\times\frac{21}{20}-x)\frac{21}{20}-x$, and at the end of the third $[(1000\times\frac{21}{20}-x)\frac{21}{20}-x]\frac{21}{20}-x$, which is zero.

Hence

$$1000\times\left|\frac{21}{20}\right|^3$$

$$=x\left(1+\frac{21}{20}+\left|\frac{21}{20}\right|^2\right)$$

$$=x\left(\frac{441+420+400}{400}\right)$$

$$=x.\frac{1261}{400};$$

$$\therefore\ x=50\times\frac{9261}{1261}$$

$$=\text{£}367\ 4s.\ 2d.\ \text{nearly.}$$

Or thus: let £x be the sum required; then the amount of an annuity of £x for three years is equal to the amount of £1000 after the same interval,

i. e. $$x\left(1+\frac{21}{20}+\left|\frac{21}{20}\right|^2\right)=1000\times\left|\frac{21}{20}\right|^3,$$

the same equation as before, giving the same result.

11. *Define a Logarithm. Prove the rules of multiplication and division of numbers by the aid of logarithms. Wherein lies the convenience of our tables being calculated to base* 10?

[Todhunter's Trigonometry, §§ 132, 135, 136, 140, 141, 142.]

12. *What are oranges a gross when* 50 *more for a sovereign lowers the price twopence a score?*

Let x = number of pence per gross that oranges cost.

Then cost of one orange in pence is $\frac{x}{144}$, and number obtained for a penny is $\frac{144}{x}$; $\therefore$ number for one sovereign $= \frac{240 \cdot 144}{x}$.

Again, if they are reduced twopence a score—that is, 14·4*d.* per gross—the number for one sovereign is $\frac{240 \cdot 144}{x - \frac{144}{10}}$.

Hence $$\frac{240 \cdot 144}{x - \frac{144}{10}} - \frac{240 \cdot 144}{x} = 50,$$

or $$\frac{240 \cdot 144 \cdot 144}{(10x - 144)x} = 50;$$

$$\therefore \quad 50x^2 - 720x = 24 \cdot 144^2;$$

$$\therefore \quad 100x^2 - 1440x = 48 \cdot 144^2;$$

$$\therefore \quad (10x - 72)^2 = 72^2(48 \cdot 4 + 1)$$

$$= 72^2 \cdot 193;$$

$$\therefore \quad 10x = 72(1 + \sqrt{193}),$$

$$= 1080 \text{ nearly};$$

$$\therefore \quad x = 108 \text{ nearly};$$

$\therefore$ cost per gross = nearly 9*s.*

Afternoon, 3 *to* 6.

GEOMETRY AND TRIGONOMETRY.

1. *Two rectilinear triangles being supposed equiangular, show that their three pairs of homologous sides, opposite to their three pairs of equivalent angles, are proportional.*

[Euclid, vi. 4.]

2. *Four rectilinear segments being supposed proportional, show that every four rectilinear figures, similar in form and similarly described on them, are also proportional.*

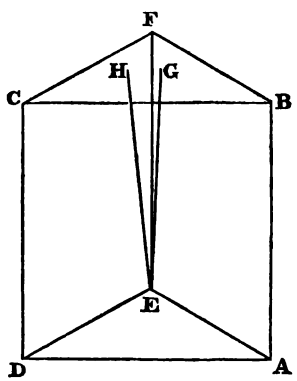
[Euclid, vi. 22.]

3. *Three planes, not having a common line of intersection, being supposed such that two of their three lines of intersection are parallel, show that the whole three are parallel.*

[Compare with Euclid, xi. 19.]

Let ABCD, CDEF, EFBA be the three given planes, AB, CD, EF their intersections, of which AB, CD are parallel; then shall EF be also parallel to AB, CD; for, if not, EF when produced must meet AB; but EF lies in the plane CDEF; and ∴ the plane CDEF contains one point in AB, and also contains the parallel line CD; hence it coincides with the plane ABCD, which contains AB, CD; but this is impossible.

Fig. 8.



∴ EF is parallel to AB,

and

∴ EF is parallel also to CD.

The theorem may also be proved as follows:—

If EF be not parallel to AB, CD, through E draw, in the plane ABEF, a line EG parallel to AB, and in the plane CDEF a line EH

parallel to CD, and therefore to AB; hence through the point E we have two lines drawn parallel to the lines AB and CD, which is impossible; that is to say EG, EH coincide with each other, and therefore with EF, since EF is the only line common to ABEF, CDEF.

Therefore EF is parallel to AB, CD.

4. *A plane being supposed to intersect a sphere, show that the intersection of the two is a circle whose centre is the foot of the perpendicular upon the plane from the centre of the sphere.*

[See Wilson's Solid Geometry, p. 52.]

5. *Three cylinders being supposed to pass through the three vertices, and to have for axes the three opposite sides of a rectilinear triangle, show that their three curved surfaces are equal in area.*

Let ABC be the triangle, and let AD be drawn perpendicular to BC; then AD is the radius of the cylinder which has BC for its axis, and which passes through A.

Its area is therefore equal to the rectangle contained by BC and the perimeter of the circle whose radius is AD; or equal to the rectangle contained by BC and 2π. AD; or 2π. rect. AD. BC; but rect. AD. BC = twice the area of the triangle.

Therefore this cylinder has an area = $4\pi \times$ (the area of the triangle).

Similarly each of the other cylinders may be shown to have this same area; hence the areas of all three are equal.

6. *A cone and hemisphere being supposed to have equal bases and altitudes, determine the ratios* (a) *of their convex surfaces,* (b) *of their entire volumes.*

Let r be the radius of their bases, then r is also their common altitude; then the slant height of the cone $= \sqrt{r^2+r^2} = r\sqrt{2}$ [see quest. 6, 1878]; and the convex surface of the cone is $\frac{1}{2}r\sqrt{2}.2\pi r = \pi r^2\sqrt{2}$; that of the

hemisphere is $2\pi r^2$: hence the ratio of their convex surfaces is $\pi r^2\sqrt{2} : 2\pi r^2$, or $1 : \sqrt{2}$.

Again, the volume of the cone is $\frac{1}{3}r \times \pi r^2 = \frac{\pi r^3}{3}$, and that of the hemisphere $= \frac{1}{2}$ that of the whole sphere $= \frac{1}{2} \cdot \frac{4}{3} \cdot \pi r^3 = \frac{2\pi r^3}{3}$.

Hence the ratio of their volumes is $\frac{\pi r^3}{3} : \frac{2\pi r^3}{3}$ or $1 : 2$.

7. *A rectilinear triangle being supposed circumscribed by a circle, show that the ratio of any side to the diameter of the circle is equal to the sine of the opposite angle.*

[Todhunter's Plane Trigonometry, § 252.]

8. *By aid of the preceding, or otherwise, express the diameter d of the circle circumscribing a rectilinear triangle, in terms of the three sides, a, b, c, of the triangle.*

This can be easily deduced as follows:—

$$\frac{a}{d} = \sin A, \quad \frac{b}{d} = \sin B, \quad \frac{c}{d} = \sin C, \quad \text{and} \quad A + B + C = 180°;$$

$$\therefore \quad \sin A = \sin (B + C)$$
$$= \sin B \cos C + \sin C \cos B;$$

$$\therefore \qquad a = b\sqrt{1 - \frac{c^2}{d^2}} + c\sqrt{1 - \frac{b^2}{d^2}}, \quad . \quad . \quad . \quad (1)$$

from which we have to find d.

Since

$$b^2 - c^2 = b^2\left(1 - \frac{c^2}{d^2}\right) - c^2\left(1 - \frac{b^2}{d^2}\right),$$

$$\therefore, \text{ from (1)}, \quad \frac{b^2 - c^2}{a} = b\sqrt{1 - \frac{c^2}{d^2}} - c\sqrt{1 - \frac{b^2}{d^2}}. \quad . \quad . \quad (2)$$

(This artifice is of frequent use in dealing with surds.)

Hence, adding (1) and (2), we get

$$\frac{a^2+b^2-c^2}{a}=2b\sqrt{1-\frac{c^2}{d^2}};$$

squaring

$$\frac{a^4+b^4+c^4+2a^2b^2-2b^2c^2-2c^2a^2}{a^2}$$

$$=4b^2-\frac{4b^2c^2}{d^2};$$

$$\therefore\quad -\frac{4a^2b^2c^2}{d^2}=a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$$

$$=-(a+b+c)(b+c-a)(c+a-b)(a+b-c).$$

If $a+b+c=2s$, this becomes

$$\frac{a^2b^2c^2}{d^2}=4s(s-a)(s-b)(s-c);$$

$$\therefore\quad d=\frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}.$$

9. *Given for three trigonometrical angles, α, β, γ, that* $\tan\alpha=a$, $\tan\beta=b$, $\tan\gamma=c$; *find the value of* $\tan(\alpha+\beta+\gamma)$ *in terms of* a, b, c.

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\,\tan\beta}=\frac{a+b}{1-ab},$$

$$\tan(\alpha+\beta+\gamma)=\frac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$$

$$=\frac{\frac{a+b}{1-ab}+c}{1-\frac{a+b}{1-ab}.c}$$

$$=\frac{a+b+c-abc}{1-bc-ca-ab}.$$

10. *Given, of a rectilinear triangle, the base* c, *and the two base-angles* A *and* B; *find, in terms of them, the two sides* a *and* b, *the altitude* h, *and the area* Δ.

Let CD be perpendicular to AB.

We have $C=180°-(A+B)$,

$$\frac{a}{c}=\frac{\sin A}{\sin C}=\frac{\sin A}{\sin(A+B)};$$

$$\therefore\quad a=\frac{c\sin A}{\sin(A+B)}.$$

Similarly $$b=\frac{c\sin B}{\sin(A+B)},$$

and $$h=a\sin B=b\sin A=\frac{c\sin A\sin B}{\sin(A+B)};$$

also $$\Delta=\tfrac{1}{2}.hc$$

$$=\frac{c^2}{2}.\frac{\sin A\sin B}{\sin(A+B)}.$$

11. *The coordinates (rectangular) of three points* P_1 Q_1 R_1 *in a plane being respectively* (3, 4), (5, 6), (7, 8), *determine, by any method, the area of the triangle* PQ .

By the usual formula the area is immediately found to be zero; or the three points lie in one straight line.

12. *The equations, in rectangular coordinates, of a circle in a plane, and of a straight line through its centre, being respectively* $x^2+y^2=r^2$ *and* $x+y=0$; *find, by any method, those of the two tangents to the circle which are parallel to the line.*

[See question 11 in the same paper for 1878.]

Any tangent to the given circle may be written

$$x\cos\phi+y\sin\phi=r.$$

But this is to be parallel to

$$x+y=0;$$

hence $\cos\phi=\sin\phi$, and each is $\pm\frac{1}{\sqrt{2}}$.

The tangents therefore are

$$x+y=\pm r\sqrt{2}.$$

Wednesday, July 21, 1880.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

Morning, 10 *to* 1.

ARITHMETIC AND ALGEBRA.

1. *Prove that a number is divisible by nine if, and not unless, the sum of its digits is divisible by nine.*

[See Todhunter's Algebra, §§ 444, 446.]

Let the digits in order from the left be $p_n, p_{n-1}, \ldots p_0$; then the number is

$$p_n 10^n + p_{n-1} 10^{n-1} + \ldots + p_1 10 + p_0$$
$$= p_n(10^n - 1) + p_{n-1}(10^{n-1} - 1) + \ldots + p_1(10 - 1)$$
$$+ p_n + p_{n-1} + p_{n-2} + \ldots + p_1 + p_0.$$

All the terms in the first line are divisible by 9; hence if the terms in the second line are also, so is the whole number, and *vice versâ*; which proves the theorem.

2. *A steamer goes* 9·6 *miles in an hour in still water. How long will it take to run* 10 *miles up a stream, and return, the velocity of the stream being two miles an hour?*

Rate up stream = (9·6 − 2) miles an hour,
= 7·6 miles an hour;

$\therefore$ time required to travel 10 miles up stream $= \frac{10}{7 \cdot 6}$ hours.

Similarly rate down stream = 11·6 miles an hour;

$\therefore$ time required to travel down $= \frac{10}{11 \cdot 6}$ hour;

$\therefore$ time required to go and return $= \left(\frac{10}{7 \cdot 6} + \frac{10}{11 \cdot 6}\right)$ hours
$= (1 \cdot 32 + \cdot 86)$ hours,
$= 2 \cdot 18$ hours.

3. *A man purchases the leasehold of a house for* £3000, *the ground-rent being* £50 *per annum, and the lease having* 20 *years to run. He lets the house for the whole term. What annual rent should he receive to make* 5 *per cent. on his outlay.* [log 2 = 0·3010300, log 3 = 0·4771213, log 7 = 0·8450980, log 11306 = 4·0533090, log 11307 = 4·0533474.]

Let £x = the annual profit which he should make; then, in order that he may gain 5 per cent. on his outlay, the present value of an annuity of £x for 20 years must be £3000 × (1·05) or £3150.

But the present value of this annuity is

$$£x\left(\frac{1}{1\cdot05}+\frac{1}{(1\cdot05)^2}+\ldots+\frac{1}{(1\cdot05)^{20}}\right)$$

or

$$£x\,\frac{\{(1\cdot05)^{20}-1\}}{(1\cdot05)^{20}(1\cdot05-1)}.$$

Hence

$$£x\,\frac{\{(1\cdot05)^{20}-1\}}{(1\cdot05)^{20}(1\cdot05-1)}=£3150;$$

we have then

$$x=\frac{3150\times\cdot05}{1-(1\cdot05)^{-20}}.$$

To find $(1\cdot05)^{-20}$ proceed thus:

$$\log(1\cdot05)=\log 3+\log 7-\log 20$$
$$=\cdot0211893;$$
$$\therefore\ \log(1\cdot05)^{-20}=-(\cdot423786).$$

To find the corresponding number from the given logarithms, we need an artifice similar to that used in question 2, page 34.

$$\cdot423786=\cdot4771213-\cdot0533353$$
$$=\log 3-\cdot0533353.$$

Hence $\log(1\cdot05)^{-20}=\cdot0533353-\log 3$.

Now ·0533353 lies between the logs of 1·1306 and

1·1307, and may be readily found to be log 1·130669. Hence

$$\log(1{\cdot}05)^{-20} = \log(1{\cdot}130669) - \log 3;$$

$$\therefore (1{\cdot}05)^{-20} = {\cdot}3768897;$$

$$\therefore 1-(1{\cdot}05)^{-20} = {\cdot}6231103;$$

$$\therefore x = \frac{157{\cdot}5}{{\cdot}6231103}$$

$$= 252{\cdot}7644.$$

But £x is the net annual profit; in order therefore to cover the ground-rent, the rent must be £50 more than this, or £302·7644,

or £302 15s. 3½d.

4. *An iron bar breaks under a tensile strain of* 21 *tons per square inch of section. What is this in grammes per square centimetre?*

[*One metre*=39⅜ *inches; a cubic foot of water weighs* 1000 *ounces; a kilogramme is the weight of a cubic decimetre of water.*]

$$\text{One metre} = \frac{315}{8} \text{ inches};$$

$$\therefore 1 \text{ inch} = \frac{8}{315} \text{ metres},$$

$$= \frac{80}{315} \text{ decimetres} = \frac{800}{315} \text{ centimetres}.$$

$$\text{Now 1 oz.} = \text{weight of } \frac{1}{1000} \text{ c. foot or } \frac{1728}{1000} \text{ c. in. of water},$$

$$= \text{weight of } \frac{1728}{1000} \times \left|\frac{80}{315}\right|^3 \text{ c. decim. of water},$$

$$= \frac{1728 \times 8^3}{315} \text{ kilogrammes}.$$

Now a strain of 21 tons per square inch is the same as

$21 \times 2240 \times 16$ ounces per square inch;

or

$$\frac{21 \cdot 2240 \cdot 16 \cdot 1728 \cdot 8^3}{315^3} \text{ kilogs. per square inch;}$$

or

$$\frac{21 \cdot 2240 \cdot 16 \cdot 1728 \cdot 8^3}{315^3} \times \left|\frac{\overline{315}}{800}\right|^2 \text{ kilogs. per sq. centim.;}$$

or

$$\frac{21 \cdot 2240 \cdot 16 \cdot 1728 \cdot 8^3 \cdot 315^2 \cdot 1000}{315^3 \cdot 800^2} \text{ grms. per sq. centim.;}$$

or

3303014·4 grammes per square centimetre.

5. *Find a and b, that*

$x^3+ax^2+11x+6$, *and* $x^3+bx^2+14x+8$

may have a common factor of the form x^2+Ax+B.

Let the other factors be respectively $x+C$, $x+D$; then clearly $CB=6$, $DB=8$, and

$$\left.\begin{aligned} x^3+ax^2+11x+6 &= \left(x+\frac{6}{B}\right)(x^2+Ax+B) \\ x^3+bx^2+14x+8 &= \left(x+\frac{8}{B}\right)(x^2+Ax+B) \end{aligned}\right\} \text{identically.}$$

Hence, equating the various coefficients,

$$\left.\begin{aligned} A+\frac{6}{B} &= a \quad ..(1), \\ B+\frac{6A}{B} &= 11 \quad ..(2), \end{aligned}\right\}$$

$$\left.\begin{aligned} A+\frac{8}{B} &= b \quad ..(3), \\ B+\frac{8A}{B} &= 14 \quad ..(4). \end{aligned}\right\}$$

Hence from (2) and (4), $B=2$, $A=3$;

and from (1) and (3), $\left.\begin{aligned} a &= 6 \\ b &= 7 \end{aligned}\right\}$.

6. *Prove that the Arithmetic mean of two numbers is greater than their Geometric mean.*

Let a, b be the numbers: then

$$\text{their A. M.} = \frac{a+b}{2}, \text{ and}$$

$$\text{their G. M.} = \sqrt{ab};$$

$$\therefore \quad \text{A. M.} - \text{G. M.} = \tfrac{1}{2}(a - 2\sqrt{ab} + b)$$

$$= \tfrac{1}{2}(\sqrt{a} - \sqrt{b})^2,$$

which is positive. $\therefore$ the A. M. is greater than the G.M.

7. *A man has in his pocket £10 in sovereigns, shillings, and pence; the total number of coins is 62, and the weight is $23\frac{1}{12}$ ounces. Assuming that (for the purposes of this question only) a sovereign weighs $\frac{1}{4}$ ounce, a shilling $\frac{1}{6}$ ounce, and a penny $\frac{1}{2}$ ounce; how many sovereigns, shillings, and pence has he respectively?*

Let x, y, z be the numbers of sovereigns, shillings, and pence respectively that the man has; then their value in pence is

$$240x + 12y + z = 2400, \text{ by question;}$$

also their weight in ounces is

$$\frac{x}{4} + \frac{y}{6} + \frac{z}{2} = 23\tfrac{1}{12};$$

and, lastly,

$$x + y + z = 62.$$

From these three simultaneous equations we find x, y, z in the usual manner: the result is

$$x = 9 = \text{number of sovereigns},$$

$$y = 17 = \text{number of shillings},$$

$$z = 36 = \text{number of pence}.$$

8. *Find the number of permutations of the letters of the word "proportion." In how many orders can you arrange ten things round the circumference of a circle?*

For the first part we require the number of permutations of 10 things where there are 2 p's, 2 r's, and 3 o's; by a well-known formula the result is

$$\frac{\lfloor 10}{\underline{2}\,\underline{2}\,\underline{3}} = 151200.$$

For the second part, fix any one thing, then the others can be arranged in $\lfloor 9$ ways from left to right, which, if there is a distinction between a distribution from left to right, and a distribution in the same order from right to left, is the result.

If, however, there is no such distinction, we must halve the above result; this gives $\frac{\lfloor 9}{2}$.

For in this case, any order from left to right, and the same order from right to left, must be counted as only one arrangement.

For example, if there were four things, A, B, C, D, the arrangements

$$\begin{matrix} & \mathrm{A} & \\ \mathrm{D} & & \mathrm{B} \\ & \mathrm{C} & \end{matrix} \quad \text{and} \quad \begin{matrix} & \mathrm{A} & \\ \mathrm{B} & & \mathrm{D} \\ & \mathrm{C} & \end{matrix}$$

are such that though each thing has the same neighbours in both, yet they are right and left hand neighbours in one case, left and right hand in the other; and similarly for any number of things.

9. *A purchaser is to take a plot of land fronting a street; the plot is to be rectangular, and three times its frontage added to twice its depth is to be* 96 *yards. What is the greatest number of square yards he may take?*

Let x be the frontage in yards,
y the depth in yards,
then xy is the area in square yards.
Now $3x+2y=96$,
and xy is to be a maximum;
that is to say, $\frac{x(96-3x)}{2}$ is to be a maximum.

Call this A, then

$$\frac{x(96-3x)}{2}=A;$$

$$\therefore\ 3x^2-96x=-2A;$$

$$\therefore\ (3x-48)^2=48^2-6A;$$

$$\therefore\ 3x=48\pm\sqrt{48^2-6A}.$$

Now x is real, and the greatest value that A can have will therefore be given by

$$6A=48^2;$$

$$\therefore\ A=\frac{48^2}{6}=384;$$

$\therefore$ maximum area is 384 square yards.

A more symmetrical way of solving the problem is as follows:—

$$3x+2y=96,$$

xy is a maximum,

$\therefore$ $6xy$ „ „

or $3x\,.\,2y$ „ „ .

Now $$3x\,.\,2y=\tfrac{1}{4}[(3x+2y)^2-(3x-2y)^2]$$
$$=\tfrac{1}{4}[96^2-(3x-2y)^2],$$

which is a maximum when

$$\left.\begin{aligned}3x=2y=48,\\ xy=384,\end{aligned}\right\}\text{ as before.}$$

10. *What are ashes per* 100 *loads when eight more loads for a sovereign lowers the price a penny a load? Of what problem is the negative answer the solution?*

Let x be the price per load in pence; then number of loads for a sovereign is $\frac{240}{x}$; but if $x-1$ were the price per load in pence, the number of loads for a sovereign would be $\frac{240}{x-1}$;

$$\therefore \quad \frac{240}{x} = \frac{240}{x-1} - 8,$$

$$\text{or} \quad \frac{30}{x-1} = \frac{30}{x} + 1;$$

$$\therefore \quad x^2 - x - 30 = 0,$$

$$\text{and} \quad x = 6 \text{ or } -5;$$

$\therefore$ price is 6*d.* per load, and £2 10*s.* per 100; or $-5d.$ per load, which means that 5*d.* will be given for removing each load, and the problem of which this is the solution stands thus:—

What is *given* per load for removing ashes, if when eight loads less are removed for a sovereign a penny per load more is obtained? The answer is 5*d.*

Afternoon, 3 *to* 6.

GEOMETRY AND TRIGONOMETRY.

1. *If four right lines be proportional, prove that the rectangle contained by the extremes is equal to the rectangle contained by the means.*

[Euclid, vi. 16.]

2. *Find a fourth proportional to three given right lines. If the first and second, instead of being lines, be given rectilinear figures, how can the fourth proportional be found?*

For the first part, see Euclid, xi. 12.

For the second part, let A, B be the given rectilineal figures, and X the given straight line: it is required to find a fourth proportional to the figures A, B and the line X.

Take any straight line PQ, and to it apply (Euclid, i. 44, 45) rectangles PR, PS, equal respectively to A and B; then their bases RQ, SQ are in the same straight line, and so are their sides opposite to their bases. Hence

figure PR : figure PS : : base QR : base QS (Euclid, vi. 1);

or figure A : figure B : : base QR : base QS.

Hence we have replaced the ratio of the given figures by the ratio of two straight lines; and we may now proceed to find a fourth proportional to the lines QR, QS, X. But if this is Y, we have

$$QR : QS :: X : Y,$$

and $$\text{fig. A} : \text{fig. B} :: QR : QS;$$

$$\therefore \quad \text{fig. A} : \text{fig. B} :: X : Y;$$

or Y is a fourth proportional to the two figures A, B and the straight line X.

3. *If a right line be perpendicular to each of two right lines at their point of intersection, prove that it is perpendicular to the plane which contains the right lines.*

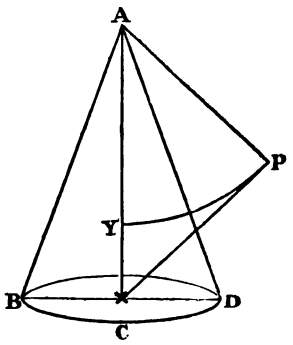
[Euclid, xi. 4.]

4. *If a plane intersect a sphere, prove that their curve of intersection is a circle; and show how to determine its centre.*

[Wilson's Solid Geometry, p. 52.]

5. *Find an expression for the lateral superficial area of a right cone; and show how to bisect the surface of a right cone by a plane drawn parallel to its base.*

Fig. 9.



[For the first part, see Wilson's Solid Geometry, p. 45.]

Let ABCD be the given right cone, A its vertex, AX its axis. Now, if any plane parallel to its base BCD cut the axis in Y, then the area cut off by this plane is to the area of the whole cone as the square on AY is to the square on AX. But we wish to cut off just half the curved area, hence this plane will be the plane required if the ratio of these squares is 1 to 2.

To determine AY, then, describe on AX as base an isosceles right-angled triangle APX, and with centre A and radius AP describe a circle cutting AX in Y; Y is the point where the plane required meets the axis.

For $\because$ APX is a right-angled triangle,

$\therefore$ sq. on AX = sqq. AP, PX,

and $\because$ AP = PX = AY;

$\therefore$ sq. on AX = 2 sq. AP
= 2 sq. AY;

$\therefore$ the sq. on AY : sq. on AX :: 1 : 2;

$\therefore$ the area cut off from the cone by the plane through Y parallel to the base is half the whole area;

$\therefore$ this plane bisects the curved area of the given right cone.

6. *Prove that the volume of a sphere is to that of its circumscribing cylinder in the proportion of* 2 *to* 3.

[See Wilson's Solid Geometry, p. 58.]

7. *Find the simplest forms of the expressions*

$$\frac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} \text{ and } \frac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta}.$$

The first expression is

$$\frac{\sin 7\theta - \sin 5\theta}{\cos 7\theta + \cos 5\theta} = \frac{2 \sin \theta \cos 6\theta}{2 \cos \theta \cos 6\theta} = \tan \theta;$$

the second is

$$\frac{\cos 6\theta - \cos 4\theta}{\sin 6\theta + \sin 4\theta} = -\frac{2 \sin \theta \sin 5\theta}{2 \cos \theta \sin 5\theta} = -\tan \theta.$$

The simplest forms are therefore $\tan\theta$ and $-\tan\theta$ respectively.

8. *Prove the formula for the area of a plane triangle in terms of its three sides.*

If the sides of a triangle be 114, 101, *and* 25 *respectively, calculate its area.*

The simplest way of finding the area of the triangle in terms of the sides, without any previous calculation, is thus:—Let ABC be the triangle; let AD, at right angles to BC, be called p; then area required is $A=\frac{p\,a}{2}$; and to find p we have

$$BD+DC=BC=a,$$

or
$$\sqrt{c^2-p^2}+\sqrt{b^2-p^2}=a;$$

$$\therefore\quad \sqrt{c^2-p^2}-\sqrt{b^2-p^2}=\frac{c^2-b^2}{a}.$$

(See Question 8 in the same paper for 1879.)

$$\therefore\quad \sqrt{c^2-p^2}=\frac{a^2+c^2-b^2}{2a};$$

$$\therefore\quad p^2=\frac{4a^2c^2-(a^2+c^2-b^2)^2}{4a^2}$$

$$=\frac{(2ac-a^2-c^2+b^2)(2ac+a^2+c^2-b^2)}{4a^2}$$

$$=\frac{[b^2-(a-c)^2][(a+c)^2-b^2]}{4a^2};$$

$$\therefore\quad \frac{p^2a^2}{4}=A^2=\frac{[b^2-(a-c)^2][(a+c)^2-b^2]}{16}$$

$$=\frac{(b+c-a)(b+a-c)(a+c+b)(a+c-b)}{16};$$

$$\therefore\quad A=\tfrac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}.$$

If we call $\frac{a+b+c}{2}$, s, this of course assumes the more usual form

$$A=\sqrt{s(s-a)(s-b)(s-c)}.$$

In the numerical example,

$$s=120,\ s-a=6,\ s-b=19,\ s-c=95;$$

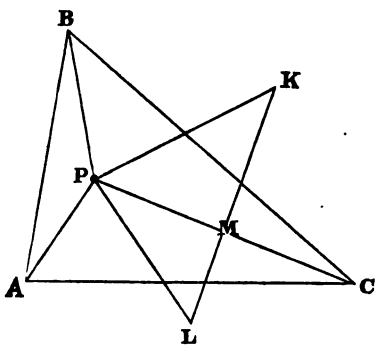
$$\therefore\ A=\sqrt{120\ .\ 6\ .\ 19\ .\ 95}$$

$$=\ 6\ .19\ .\ 5\ .\ 2=1140.$$

9. *The lengths of the lines joining three points* A, B, C *are observed. At any point* P, *in the plane* ABC, *the angles* APC *and* BPC *are observed. Show how the distances from* P *to the points* A, B, C *can be determined.*

Let K, L be the centres of the circles passing through B, P, C; A, P, C respectively; join KL; then, since PC is

Fig. 10.



the common chord of the two circles, it will be bisected at right angles by KL; join also PK, PL.

Let $\angle BPC=\alpha$, $\angle APC=\beta$, and $\angle BPA=2\pi-\alpha-\beta=\gamma$.

Now $\angle PKL=$ half $\angle PKC$

$=\angle PBC$ (Euclid, iii. 20),

and, similarly, $\angle PLK=\angle PAC$.

Hence $\angle PLK+\angle PKL=\angle PBC+\angle PAC$

$=2\pi-\alpha-\beta-C;$

$\therefore\ \angle LPK=\alpha+\beta+C-\pi.$

Now,

$$\text{area of PKL is } \tfrac{1}{2} \cdot \text{PK} \cdot \text{PL} \sin \text{LPK} = \tfrac{1}{2}\text{PM} \cdot \text{LK},$$

$$\therefore \quad \text{PM} = \frac{\text{PK} \cdot \text{PL} \sin \text{LPK}}{\text{LK}};$$

but PK = radius of circle BPC

$$= \frac{\text{BC}}{2 \sin \alpha} = \frac{a}{2 \sin \alpha},$$

similarly,

$$\text{PL} = \frac{b}{2 \sin \beta};$$

and

$$\text{LK} = \sqrt{\text{PK}^2 + \text{PL}^2 - 2\text{PK} \cdot \text{PL} \cos \text{LPK}}$$

$$= \tfrac{1}{2}\sqrt{\frac{a^2}{\sin^2\alpha} + \frac{b^2}{\sin^2\beta} + \frac{2ab}{\sin\alpha \sin\beta} \cos(\alpha+\beta+\text{C})};$$

$$\therefore \quad \text{PM} = \frac{\pm ab \sin(\alpha+\beta+\text{C})}{2\sqrt{a^2 \sin^2\beta + b^2 \sin^2\alpha + 2ab \sin\alpha \sin\beta \cos(\alpha+\beta+\text{C})}},$$

taking the sign so that PM is positive. And PC = 2PM. Similar formulæ hold for PA and PB.

The same result can be obtained by different methods; but this one is interesting because it shows *how* the result comes to take the form given above.

10. *Being given the bases and the sum of the areas of a number of triangles having a common vertex, show how to find the locus of the vertex by Coordinate Geometry.*

Let the equations to the lines in which the bases lie be expressed in theform

$$x \cos a_1 + y \sin a_1 - p_1 = 0, \quad \ldots\ldots \quad (1)$$

$$x \cos a_2 + y \sin a_2 - p_2 = 0, \quad \ldots\ldots \quad (2)$$

&c.,

and let the lengths of the bases be a_1, a_2, a_3, &c. Then the area of any triangle of the system is equal to half the rectangle contained by the perpendicular from its vertex on its base and the base.

Let X, Y be the vertex; then the perpendicular from X, Y on (1) is

$$X\cos\alpha_1+Y\sin\alpha_1-p_1.$$

Hence the area of the triangle whose base is in this line is

$$\tfrac{1}{2}a_1(X\cos\alpha_1+Y\sin\alpha_1-p_1).$$

Similarly for the areas of the other triangles. Hence the sum of the areas is

$$\tfrac{1}{2}\{a_1(X\cos\alpha_1+Y\sin\alpha_1-p_1)+a_2(X\cos\alpha_2+Y\sin\alpha_2-p_2)+\&c.\}$$

$$=\text{a given quantity}=U.$$

Hence we have for the locus of (X, Y) a straight line.

This line may also be written, with the usual abbreviations,

$$X\Sigma a\cos\alpha+Y\Sigma a\sin\alpha=\Sigma ap+2U.$$

Note. By the "sum of the areas" is meant the algebraic sum of the areas; otherwise we should, as the locus crossed each straight line, have a discontinuity, and the locus would consist of portions of different straight lines joined at their extremities.

11. *Find the equations of the right lines which bisect the angles between the lines*

$$2x+3y-5=0,\quad 3x+2y-7=0.$$

If (X, Y) is on either of the bisectors, the perpendiculars from X, Y on the lines are equal in length, or

$$\frac{2X+3Y-5}{\sqrt{2^2+3^2}}=\pm\frac{3X+2Y-7}{\sqrt{3^2+2^2}},$$

these being the expressions for these perpendiculars.

This reduces at once to

$$2X+3Y-5=\pm(3X+2Y-7);$$

hence

$$\begin{cases} X+Y=\frac{1}{5}^2, \\ \text{or} \\ X-Y=2. \end{cases}$$

These are the equations one of which X, Y must satisfy, *i. e.* the equations to the bisectors required.

12. *Show that the locus of a point from which the tangents to two circles are of equal length is a right line; and determine its equation when the equations of the circles are given.*

Taking the equations of the circles in their most general form with rectangular coordinates, we have, say,

$$(x-a_1)^2+(y-b_1)^2=c_1^2,$$
$$(x-a_2)^2+(y-b_2)^2=c_2^2.$$

Now, if we draw the tangent to the first circle from P, (X, Y), it, the line joining P to the centre, and the radius to the point of contact, form a right-angled triangle; in other words, if O be the centre and Q the point of contact,

$$OQ^2+PQ^2=OP^2;$$

or

$$PQ^2=OP^2-OQ^2,$$
$$=(X-a_1)^2+(Y-b_1)^2-c_1^2.$$

Similarly the (tangent)^2 to the other circle is

$$(X-a_2)^2+(Y-b_2)^2-c_2^2;$$

but these are equal; hence

$$(X-a_1)^2+(Y-b_1)^2-c_1^2=(X-a_2)^2+(Y-b_2)^2-c_2^2,$$

or

$$2X(a_1-a_2)+2Y(b_1-b_2)=a_1^2-a_2^2+b_1^2-b_2^2-c_1^2+c_2^2,$$

which, being of the first degree in X, Y, shows that (X, Y) lies on a fixed straight line.

Wednesday, July 20, 1881.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

Morning, 10 *to* 1.

ARITHMETIC AND ALGEBRA.

1. *Supposing that a steam-tug travels* 10 *miles an hour in still water when alone, but draws a barge* 4 *miles an hour. It has to take a barge* 10 *miles up a stream which runs* 1 *mile*

an hour, and then to return without the barge. How long will it take for the journey?

Rate, with barge, against stream, is 3 miles an hour.

$\therefore$ time up stream $= \frac{10}{3}$ hours.

Rate, without barge, with stream, is 11 miles an hour.

$\therefore$ time down stream $= \frac{10}{11}$ hours.

$\therefore$ whole time $= \frac{10}{3} + \frac{10}{11}$ hours,

$= 4\frac{8}{33}$ hours.

$= 4$ h . $14\frac{6}{11}$ minutes.

2. *Reduce the following to a vulgar fraction reduced to its simplest form:—*

$$\frac{0\cdot\dot{5} \times 1\cdot\dot{7}14285\dot{} \times 1\cdot0\dot{7}69230\dot{}}{2\cdot\dot{8}57142\dot{} \times 2\cdot\dot{3}07692\dot{} \times 14\cdot\dot{5}}.$$

$$\cdot\dot{5} = \tfrac{5}{9},$$

$$1\cdot\dot{7}14285\dot{} = 1\tfrac{5}{7} = \tfrac{12}{7},$$

$$1\cdot0\dot{7}69230\dot{} = 1\tfrac{769230}{999999} = 1\tfrac{1}{13} = \tfrac{14}{13},$$

$$2\cdot\dot{8}57142\dot{} = 2\tfrac{6}{7} = \tfrac{20}{7},$$

$$2\cdot\dot{3}07692\dot{} = 2\tfrac{4}{13} = \tfrac{30}{13},$$

$$14\cdot\dot{5} = 14\tfrac{5}{9} = \tfrac{131}{9}.$$

The fraction is therefore

$$\frac{\frac{5}{9} \cdot \frac{12}{7} \cdot \frac{14}{13}}{\frac{131}{9} \cdot \frac{20}{7} \cdot \frac{30}{13}} = \frac{5 \cdot 12 \cdot 14}{131 \cdot 20 \cdot 30} = \frac{7}{655}.$$

3. *How many years will it take £100 to accumulate to £1000 at 4 per cent. compound interest?* [log 2 = 0·3010300; log 13 = 1·1139434.]

Here we have $(1\cdot04)^x 100 = 1000$, where x is the number of years required;

$$\therefore \quad x \log (1\cdot04) = \log 10 = 1,$$

or $$x[1\cdot1139434 + \cdot90309 - 2] = 1,$$

$$\therefore \quad x = \frac{1}{\cdot0170334} = 58\cdot71.$$

4. *Find a, b, c so that both $x^4+ax^3+bx^2+cx+1$ and $x^4+2ax^3+2bx^2+2cx+1$ are perfect squares.*

Let the square roots be respectively x^2+px+q, and x^2+rx+s. Squaring, and comparing coefficients, we have

$$\left.\begin{aligned} 2p&=a,\\ 2q+p^2&=b,\\ 2pq&=c,\\ q^2&=1, \end{aligned}\right\}\text{A.} \qquad \left.\begin{aligned} 2r&=2a,\\ 2s+r^2&=2b,\\ 2rs&=2c,\\ s^2&=1. \end{aligned}\right\}\text{B.}$$

If $q=+1$, since $\left.\begin{aligned} 2p&=a,\\ \text{and } 2pq&=c, \end{aligned}\right\}$

$$\therefore \quad a=c;$$

and $\therefore$ from B, s also $=+1$;

and
$$p^2+2q=\frac{a^2}{4}+2=b, \quad \ldots \quad (1)$$

$$r^2+2s=a^2+2=2b; \quad \ldots \quad (2)$$

$\therefore$ from (1) and (2) $\quad b=3, \quad a=\pm 2=c;$

and the roots are

$$(x^2\pm x+1), \quad (x^2\pm 2x+1).$$

Again, if $q=-1$, then $a=-c=\pm 2$, and $b=1$, and our roots become

$$(x^2\pm x-1), \quad (x^2\pm 2x-1).$$

The values of a, b, c are therefore

$$\begin{aligned} a&=+2, \quad b=3, \quad c=+2,\\ a&=-2, \quad b=3, \quad c=-2,\\ a&=+2, \quad b=1, \quad c=-2,\\ a&=-2, \quad b=1, \quad c=+2. \end{aligned}$$

5. *How many words of three letters in which the first and third letters are consonants and the second a vowel, but in which the first and third letters are not the same, can be formed with an alphabet of 20 consonants and 5 vowels?*

Number of permutations of two consonants $=20\,.\,19$.

But between every pair any of the five vowels may be placed.

$\therefore$ number of words $=20\,.\,19\times 5$

$$=1900.$$

6. *Assume the House of Commons to consist of Liberals, Conservatives, and Home Rulers. In one division half the Liberals,* 100 *Conservatives, and two thirds of the Home Rulers were absent. The Liberals voted with, the Conservatives and Home Rulers against, Government, which had a majority of five. On another occasion, all the members voting, the Liberals had a majority of* 40 *over the Conservatives and Home Rulers combined. On a vote of urgency,* 75 *Liberals, and half the Conservatives and* 10 *Home Rulers being absent, the combined Liberals and Conservatives told eight times as many votes as the Home Rulers. How many Members of Parliament were there?*

Let x = number of Liberals,

y = number of Conservatives,

z = number of Home Rulers;

then, by the question, $\frac{x}{2} = y - 100 + \frac{z}{3} + 5$,

$$x = y + z + 40,$$

$$x - 75 + \frac{y}{2} = 8(z - 10);$$

hence

$$3x - 6y - 2z = -570, \quad \ldots\ldots \quad (1)$$

$$x - y - z = 40, \quad \ldots\ldots \quad (2)$$

$$2x + y - 16z = -10. \quad \ldots\ldots \quad (3)$$

From (1) and (2), $3x - 4z = 810$,

(2) and (3), $3x - 17z = 30$;

$$\therefore \quad 13z = 780,$$

$$z = 60,$$

$$x = 350,$$

$$y = 250.$$

Ans. Total number of members is 660.

7. *Find x from the equation*

$$(a^{\frac{1}{3}}+x^{\frac{1}{3}})^{\frac{1}{3}}=(a^{\frac{1}{3}}+x^{\frac{1}{3}})^{\frac{1}{2}}.$$

Raise both sides to the 6th power; we get

$$(a^{\frac{1}{3}}+x^{\frac{1}{3}})^2=(a^{\frac{1}{3}}+x^{\frac{1}{3}})^3;$$

$$\therefore \quad 2a^{\frac{1}{3}}x^{\frac{1}{3}}=3a^{\frac{2}{3}}x^{\frac{1}{3}}+3a^{\frac{1}{3}}x^{\frac{2}{3}};$$

$$\therefore \quad x^{\frac{1}{3}}=0, \text{ and } x=0;$$

or $$2a^{\frac{1}{3}}x^{\frac{1}{3}}=3(a^{\frac{1}{3}}+x^{\frac{1}{3}}).$$

$$\therefore \quad x^{\frac{1}{3}}-\frac{2x^{\frac{1}{3}}}{3}.a^{\frac{1}{3}}+\tfrac{1}{9}a^{\frac{1}{3}}=-\tfrac{8}{9}a^{\frac{1}{3}};$$

$$\therefore \quad x^{\frac{1}{3}}=\frac{a^{\frac{1}{3}}}{3}(1\pm\sqrt{-8});$$

$$\therefore \quad x^{\frac{1}{3}}=\frac{a^{\frac{1}{3}}}{27}(-23\mp5\sqrt{-8});$$

$$\therefore \quad x=\frac{a}{729}(329\pm230\sqrt{-8}).$$

8. *Find two numbers such that their sum may be* 24, *and the sum of their cubes* 5256.

Let x, y be the numbers; then

$$x^3+y^3=5256,$$
$$x+y=24;$$

$$\therefore \quad x^2-xy+y^2=219;$$

but $$x^2+2xy+y^2=576; \quad \therefore \ xy=119;$$

$$\therefore \quad (x-y)^2=100,$$
$$x-y=\pm10,$$
$$\therefore \quad x=17, 7; \ y=7, 17.$$

9. *Sum the series*

$$1+2^2x+3^2x^2+\ldots+\overline{n+1}|^2x^n.$$

This is a recurring series, and may be summed in the usual way for such series; or thus:—

Let $S=1+2^2x+3^2x^2+\ldots+\overline{n+1}|^2x^n,$

then $xS= \quad x+2^2x^2+\ldots+n^2x^n+\overline{n+1}|^2x^{n+1};$

$$\therefore S(1-x)=1+3x+5x^2+\ldots$$
$$+\overline{2n+1}|\,x^n-\overline{n+1}|^2x^{n+1},$$

and $xS(1-x)=x+3x^2+\ldots+\overline{2n-1}|\,x^n$
$$+\overline{2n+1}|\,x^{n+1}$$
$$-\overline{n+1}|^2\,x^{n+2};$$

$$\therefore S(1-x)^2=1+2x+2x^2+\ldots+2x^n$$
$$-(n^2+4n+2)x^{n+1}$$
$$+\overline{n+1}|^2x^{n+2},$$

$$=\frac{2(1-x^{n+1})}{1-x}-1$$
$$-(n^2+4n+2)x^{n+1}$$
$$+\overline{n+1}|^2x^{n+2}.$$

Hence $S(1-x)^3=1+x-x^{n+1}(n+2)^2$
$$+x^{n+2}(2n^2+6n+3)$$
$$-x^{n+3}(n+1)^2.$$

10. *What is the present worth of a perpetual annuity of £10 payable at the end of the first year, £20 at the end of the second, £30 at the end of the third, and so on, increasing £10 each year, interest being taken at 5 per cent. per annum.*

Present value of £10 due 1 year hence $= £\frac{10}{1{\cdot}05}$,

„ „ £20 „ 2 years „ $= £\frac{20}{(1{\cdot}05)^2}$,

and so on.

∴ Present value of the annuity in pounds

$$= x = \frac{10}{1{\cdot}05} + \frac{20}{(1{\cdot}05)^2} + \frac{30}{(1{\cdot}05)^3} + \frac{40}{(1{\cdot}05)^4} + \ldots$$

Call $\frac{10}{1{\cdot}05}\ a$, and $\frac{1}{1{\cdot}05}\ r$,

then $$x = a + 2ar + 3ar^2 + 4ar^3 + \ldots;$$

$$\therefore\ x(1-r) = a + ar + ar^2 + \ldots \text{to } \infty$$

$$= \frac{a}{1-r},$$

and $$x = \frac{a}{(1-r)^2} = \frac{10}{1{\cdot}05 \times {\cdot}0025}$$

$$= \frac{4000}{1{\cdot}05} = \frac{80000}{21}$$

$$= 3809{\cdot}\dot{5}2380\dot{9}.$$

∴ Value = £3809 10*s.* $5\frac{3}{4}$*d.*

Afternoon, 3 *to* 6.

GEOMETRY AND TRIGONOMETRY.

1. *Prove that two rectangles have to one another a ratio which is compounded of the ratios of their sides.*

[A particular case of Euclid, vi. 23.]

2. *Describe a rectilinear figure which shall be similar to a given rectilinear figure, and equal to five times its area.*

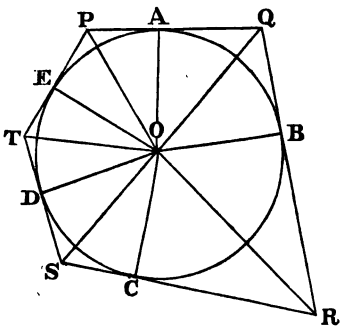
[A particular case of Euclid, vi. 25.]

3. *If the sides of a rectilinear figure all touch the same circle, prove that the area of the figure is equal to the rectangle under its semiperimeter and the radius of the circle.*

Hence show that the area of a circle is equal to the rectangle under its radius and its semicircumference.

Let PQRS . . . be the circumscribing figure, A the point of contact of PQ, B of QR, &c., and let O be the centre of the circle. Join OP, OQ, . . . OA, OB, . . . , then the area

Fig. 12.



of the triangle OPQ is half the rectangle OA, PQ; for OA is perpendicular to PQ. Similarly for all the other triangles. Therefore the area of the polygon is the sum of the areas of the triangles, or is equal to half the rectangle contained by the radius and the perimeter, since the height of each triangle is a radius of the circle.

Now increase the number of sides indefinitely, and diminish their lengths without limit, then the polygon ultimately is equal in area and perimeter to the circle, and the proposition still holds. Hence the area of a circle is equal to the rectangle contained by the radius and semicircumference.

4. *Draw a perpendicular to a given plane from a point which does not lie in the plane.*

[Euclid, xi. 11.]

5. *Find an expression for the area of the portion of the surface of a sphere cut off by any plane section; and show how to divide the surface of a sphere into any given number of equal parts by drawing parallel planes.*

[Wilson's Solid Geometry, p. 58, cor. 1.]

Since the area of a zone, or segment, of a sphere, is proportional to its height, it is clear that if in a sphere of radius a, a plane be drawn at a distance h from the centre, the area of the segment cut off is to the area of the whole sphere as the height of the segment $(a-h)$ is to the height of the sphere $(2a)$. Hence the expression for the area of the segment is

$$\frac{a-h}{2a} \times \text{area of sphere} = \frac{a-h}{2a} \cdot 4\pi a^2$$
$$= 2a\pi(a-h).$$

To divide the surface into any number of equal parts we have the following construction. Divide a diameter into the given number of equal parts, and draw through the points of division planes perpendicular to their diameter; they will divide the surface of the sphere into zones and segments of equal heights, and therefore of equal areas, which is the thing required.

6. *Being given the height of a right cone and the radius of its base, show how to construct a circle whose area shall be equal to the entire superficial area of the cone.*

Let AB be the slant side of the cone, O the centre of the base, BO the radius of the base; then, knowing AO and BO, we can at once find AB. The area of the curved surface of the cone is

$\frac{1}{2}$ rectangle (AB . circumf. of base),

and that of the base is

$\frac{1}{2}$ rectangle (OB . circumf. of base).

Hence the whole area is

$\frac{1}{2}$ rectangle [(OB + AB) . circumf. of base].

$\therefore$ whole area of cone : area of base.

:: OB+AB : OB. . . . . (1)

Take a line X, a mean proportional between OB+AB

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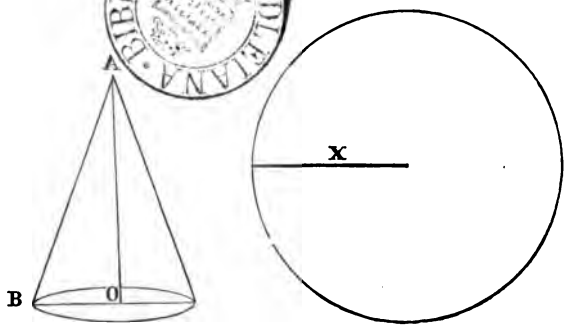
Fig. 13.



and OB, then the ratio of OB to OB+AB is the duplicate ratio of OB to X.

Now area of circle of radius OB : area of circle of radius X, in the duplicate ratio of OB to X, or in the ratio of OB to OB+AB, or in the ratio of the area of the base to the whole area of the cone, by (1); but the base is a circle of radius OB,

$\therefore$ area of circle of radius X = whole area of cone.

Hence we have described the circle required.

The analytical solution is much shorter, being simply as follows:—

Let h = height, a = radius of base } of the cone.

Then the area is $\pi a(\sqrt{h^2+a^2}+a)$;

and if x is the radius of the circle required,

$$\pi x^2 = \pi a(\sqrt{h^2+a^2}+a),$$

or $$x = \sqrt{a(\sqrt{h^2+a^2}+a^2)}.$$

7. *If* $\tan A=\frac{1}{2}$ *and* $\tan B=\frac{1}{3}$, *find the values of* $\tan(2A+B)$ *and* $\tan(2A-B)$.

$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}=\frac{4}{3};$$

$$\therefore\quad \tan(2A\pm B)=\frac{\tan 2A\pm\tan B}{1\mp\tan 2A\tan B}$$

$$=\frac{\frac{4}{3}\pm\frac{1}{3}}{1\mp\frac{4}{3}\cdot\frac{1}{3}}=\frac{\frac{5}{3}}{\frac{6}{9}},\text{ or }\frac{1}{\frac{13}{9}};$$

$$\therefore\quad \tan(2A+B)=3,$$

$$\tan(2A-B)=\frac{9}{13}.$$

8. *In a plane triangle prove the formula*

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

and write down the corresponding logarithmic equation.

$$\frac{\cos A}{1}=\frac{b^2+c^2-a^2}{2bc};$$

$$\therefore\quad \tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}=\frac{2bc-b^2-c^2+a^2}{2bc+b^2+c^2-a^2}$$

$$=\frac{a^2-(b-c)^2}{(b+c)^2-a^2}$$

$$=\frac{(a-b+c)(a+b-c)}{(a+b+c)(b+c-a)}$$

$$=\frac{(s-b)(s-c)}{s(s-a)};$$

$$\therefore\quad \tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

The logarithmic equation is

$$\mathrm{L}\tan\frac{A}{2}=\tfrac{1}{2}\Big(\log(s-b)+\log(s-c)-\log s-\log(s-a)\Big)+10.$$

9. *Explain the method of calculating the numerical sine of a small angle, and find the value of* $\sin 1°$ *to 6 decimal places.*

[For the first part, see Todhunter's Plane Trigonometry, §§ 120–130.]

For the second, the circular measure of 1° is

$$\frac{3{\cdot}14159}{180}={\cdot}0174533\ldots.=\theta,$$

$$\theta^3={\cdot}0000053\ldots.;$$

$$\therefore\ \frac{\theta^3}{6}={\cdot}000001\ldots. \text{ to 6 decimals;}$$

$$\therefore\ \sin\theta={\cdot}017452\ldots. \text{ to 6 decimals.}$$

10. *Find the equation of the right line which joins the point* (3, 2) *to the intersection of the lines* $2x+3y=1$ *and* $3x-4y=6$.

The equation

$$\lambda(2x+3y-1)+\mu(3x-4y-6)=0$$

represents a straight line. It is also satisfied by the co-ordinates of the intersection of the two given straight lines; for these make each bracketed expression to vanish. It will also pass through (3, 2) if

$$\lambda(6+6-1)+\mu(9-8-6)=0,$$

or
$$11\lambda=5\mu,$$

or
$$\frac{\lambda}{5}=\frac{\mu}{11};$$

and the line required will be

$$5(2x+3y-1)+11(3x-4y-6)=0,$$

or
$$43x-29y=71.$$

11. *Find by any method, the locus of a point being given, the sum of the squares of its distances from a number of fixed points.*

Let $(x_1 y_1)$, $(x_2 y_2)$, $(x_3 y_3)$, &c. be the fixed points, (X, Y) the point whose locus is required; then

$$(X-x_1)^2+(Y-y_1)^2+(X-x_2)^2+(Y-y_2)^2$$
$$+(X-x_3)^2+(Y-y_3)^2+ \ldots . + \ldots .$$
$$=\text{constant}=c^2,$$

the coordinates being supposed rectangular.

Or
$$n(X^2+Y^2)-2X(x_1+x_2+ \ldots . +x_n)$$
$$-2Y(y_1+y_2+ \ldots . +y_n)$$
$$+x_1^2+x_2^2+ \ldots . +y_1^2+ \ldots . =c^2,$$

which is a circle with its centre at the centre of mass of the n points.

12. *The equation of a circle in rectangular coordinates is*

$$x^2+y^2+lx+my+n=0.$$

Find the coordinates of its centre and the length of its radius. Also determine the conditions, in order that the equation

$$ax^2+bxy+cy^2+lx+my+n=0$$

should represent a circle the axes being oblique.

In the first circle let the coordinates of the centre be a, b, and the radius r; then its equation must be identically the same as

$$(x-a)^2+(y-b)^2-r^2=0.$$

Comparing the coefficients with those of

$$x^2+y^2+lx+my+n=0,$$

we find

$$-2a=l,$$
$$-2b=m,$$
$$a^2+b^2-r^2=n;$$

or
$$a=-\frac{l}{2},$$
$$b=-\frac{m}{2},$$
$$r=\tfrac{1}{2}\sqrt{l^2+m^2-4n}.$$

In the second case, if ω is the angle between the axes, h, k the coordinates of the centre, r the radius, the equation must coincide with

$$(x-h)^2+(y-k)^2+2(x-h)(y-k)\cos\omega-r^2=0.$$

Comparing coefficients,

$$\frac{1}{a}=\frac{2\cos\omega}{b}=\frac{1}{c}=\frac{-2(h+k\cos\omega)}{l}$$
$$=\frac{-2(k+h\cos\omega)}{m}$$
$$=\frac{h^2+k^2+2hk\cos\omega-r^2}{n},$$

giving
$$a=c=\frac{b}{2\cos\omega}, \quad \ldots\ldots \quad (1)$$

and three independent equations, to find h, k, r. The necessary conditions that the curve should be a circle are given by equations (1).

SECOND B.A. PASS EXAMINATION, 1881.

Wednesday, October 26, 1881.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

Morning, 10 *to* 1.

ALGEBRA AND TRIGONOMETRY.

1. *Solve the equation*

$$(x+a+b)(x+b+c)=(x+b-3a)(2x+2b-3a-c).$$

This becomes, on multiplying out and simplifying,

$$x^2-x(10a-2b+2c)+(9a^2-10ab+b^2+2ac-2bc)=0,$$

which may be solved in the ordinary manner. Or we may determine the factors of the expression on the left by examining the last term, thus:—

$$9a^2-10ab+b^2+2ac-2bc=(a-b)(9a-b+2c);$$

and the factors are

$$x-(a-b),\quad x-(9a-b+2c).$$

The roots of the given equation are therefore

$$x=a-b,\quad \text{or}\quad 9a-b+2c.$$

2. *Define a logarithm, and prove that* $\log(x^n)=n\log x$ *for all real values of n integral or fractional.*

[Todhunter's Algebra, §§ 531, 537.]

3. *Find the simplest form of the expression*

$$\frac{a^4(b-c)+b^4(c-a)+c^4(a-b)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}.$$

We find that if $b=c$, or $c=a$, or $a=b$, the numerator and denominator of this fraction vanish; $b-c$, $c-a$, and

$a-b$ are therefore factors of both. Dividing them successively by $b-c$, $c-a$, $a-b$, the expression becomes

$$\frac{-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)}{-(b-c)(c-a)(a-b)},$$

or

$$a^2+b^2+c^2+bc+ca+ab,$$

the simplest form required.

4. *If a series of numbers be in Harmonic progression, prove that their reciprocals are in Arithmetic progression.*

This theorem, which is sometimes taken as the defining property of numbers in Harmonic progression, is proved in Todhunter's Algebra, § 475.

5. *Find the sum of n terms of the series*

$$1 \cdot 2^2+2 \cdot 3^2+3 \cdot 4^2+ \ldots \text{ \&c.}$$

This series is, to n terms,

$$\begin{aligned}
&(2-1)2^2+(3-1)3^2+ \ldots . +(n+1-1)(n+1)^2 \\
&=2^3+3^3+ \ldots +(n+1)^3-[2^2+3^2+ \ldots +(n+1)^2] \\
&=1^3+2^3+ \ldots +(n+1)^3-[1^2+2^2+ \ldots +(n+1)^2] \\
&=\frac{(n+1)^2(n+2)^2}{4}-\frac{(n+1)(n+2)(2n+3)}{6}
\end{aligned}$$

[see Todhunter's Algebra, §§ 460, 461]

$$=\frac{n(n+1)(n+2)(3n+5)}{12}.$$

6. *Solve the equation*

$$\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c}=\frac{\pi}{2}.$$

Take the tangents of both sides of this equation; then (see question 9, page 44),

$$\frac{\frac{x}{a}+\frac{x}{b}+\frac{x}{c}-\frac{x^3}{abc}}{1-\frac{x^2}{bc}-\frac{x^2}{ca}-\frac{x^2}{ab}} = \text{infinity}.$$

Thus the numerator of the above fraction must be infinite, or its denominator zero. In the first case, $x=\infty$. In the second,

$$x=\pm\sqrt{\frac{abc}{a+b+c}}.$$

7. *Prove that the sines of the sides of a spherical triangle are proportional to the sines of the opposite angles.*

[Todhunter's Spherical Trigonometry, § 42.]

8. *If* $\tan A=\sqrt{3}$, $\tan B=2-\sqrt{3}$, *find the value of* $\tan(A+2B)$.

First find $\tan(A+B)$ in the usual manner: its value is $2+\sqrt{3}$. Next find $\tan(A+2B)$, by putting it in the form $\tan(\overline{A+B}+B)$; its value is infinite.

This result may be also obtained thus:—

$$\because \quad \tan A=\sqrt{3}=\tan 60^\circ,$$
$$\therefore \quad A=n\,180^\circ+60^\circ;$$

and

$$\because \quad \tan B=2-\sqrt{3}=\tan 15^\circ,$$
$$\therefore \quad B=m\,180^\circ+15^\circ;$$
$$\therefore \quad A+2B=(n+2m)\,180^\circ+90^\circ,$$

and

$$\tan(A+2B)=\tan 90^\circ=\infty.$$

9. *In a spherical right-angled triangle prove the formulæ:—*

$$\cos c=\cos a\cos b;$$
$$\tan a=\tan A\sin b.$$

[Todhunter's Spherical Trigonometry, § 67.]

10. *Show how to solve a spherical triangle when the three sides are given; also when its angles are given; and point out the connexion between the two problems.*

[Todhunter's Spherical Trigonometry, §§ 28, 48, 80, 81.]

Afternoon, 3 to 6.

GEOMETRY AND CONIC SECTIONS.

1. *Find the equation of a right line passing through the point* (2, 3) *and perpendicular to the line* $4x-3y=10$.

Since the required line is perpendicular to the line $4x-3y=10$, its equation must be of the form $3x+4y=c$; and because it also passes through the point (2, 3) we must have

$$3 \cdot 2+4 \cdot 3=c=18.$$

Hence the line required is

$$3x+4y=18.$$

2. *Given, of a parabola, its focus and two points on the curve, find the position of its vertex.*

Let S be the given focus, P, Q the given points. With centres P, Q and radii PS, QS, describe circles. Draw a common tangent to meet these circles in M, N; then MN is clearly the directrix of one of the two parabolæ satisfying the given conditions.

From S draw SX at right angles to MN; bisect SX in A: then A is clearly the vertex of the curve.

3. *In an ellipse, prove that the parallelogram formed by joining the extremities of a pair of conjugate diameters has a constant area.*

[Besant's Conic Sections, § 81.]

4. *Being given of a triangle its vertical angle in position, and the sum of n times one side, and m times the other, find the locus of the centre of its circumscribed circle.*

Let the sides OA, OB containing the given angle α be the axes of x and y; also suppose that $OA=a$, $OB=b$, and let P be the centre of the circumscribing circle; PD, PE perpendicular, and PM, PL parallel, to OA, OB respectively. Then, from the given condition,

$$ma+nb=c=\text{a constant.}$$

But $\frac{1}{2}a = \mathrm{OD} = \mathrm{OL} + \mathrm{LD} = x + y \sin \alpha$;

and $\frac{1}{2}b = \mathrm{OE} = \mathrm{OM} + \mathrm{ME} = y + x \sin \alpha$.

Hence $m(x + y \sin \alpha) + n(y + x \sin \alpha)$

$= \frac{1}{2}(ma + nb) = \frac{1}{2}c$

gives the equation to the locus required, and represents a straight line.

5. *If a chord of a curve of the second degree pass through a fixed point, prove that the locus of the point of intersection of the tangents at its extremities is a right line.*

For a geometrical proof see Besant's Conic Sections, § 192. An analytical proof may be given thus:—Let the fixed point be taken as origin, and let the conic be

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0.$$

Let (X, Y) be a point whose chord of contact passes through the origin; this chord is

$$(a\mathrm{X} + b\mathrm{Y} + g)x + (h\mathrm{X} + b\mathrm{Y} + f)y + (g\mathrm{X} + f\mathrm{Y} + c) = 0.$$

But, since it passes through the origin,

$$g\mathrm{X} + f\mathrm{Y} + c = 0;$$

and therefore (X, Y) lies on the fixed right line, whose equation is

$$gx + fy + c = 0.$$

6. *Find the condition that the right line $lx + my = n$ should touch the conic $ax^2 + 2hxy + by^2 = c$.*

Let (X, Y) be the point of contact; then the tangent at (X, Y) is

$$(a\mathrm{X} + h\mathrm{Y})x + (h\mathrm{X} + b\mathrm{Y})y = c.$$

But this is to be the same as

$$lx + my = n.$$

Hence, comparing coefficients,

$$\frac{aX+hY}{l}=\frac{hX+bY}{m}=\frac{c}{n}.$$

Thus

$$n(aX+hY)=lc,$$
$$n(hX+bY)=mc;$$

and

$$\left.\begin{aligned} n(ab-h^2)X&=c(lb-mh),\\ n(ab-h^2)Y&=c(ma-lh).\end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

But (X, Y) is also on the line $lx+my=n$,

$$\therefore \quad lX+mY=n.$$

Substituting then the values of X and Y from (1), we have

$$lc(lb-mh)+mc(ma-lh)=n^2(ab-h^2),$$

or

$$l^2bc-2\,lmch+m^2ac=n^2(ab-h^2),$$

the condition required.

7. *Being given of an ellipse a focus, a tangent in position, and the length of the minor axis, find the locus of its centre.*

Let S be the given focus, H the other focus, SY, HZ perpendiculars from S, H, on the given tangent YZ. Then,

$\because$ rect. SY . HZ = sq. on semiaxis minor = constant,

and SY is also constant,

$$\therefore \text{ HZ is constant,}$$

and the locus of H is a straight line parallel to YZ. But C, the centre, bisects SH; therefore C lies on a straight line also parallel to the tangent YZ, and at a distance from it equal to $\frac{1}{2}$HZ.

8. *Find an expression for the area of the portion of a spherical surface which is cut off by a given plane; and describe a circle whose area shall be equal to the part of the surface included between two parallel planes.*

Let AB be the diameter through the highest point, A, of

the segment, H the point where it meets the given plane, P any point in the plane and the sphere. Then, if $AB=2r$, $AH=h$, the curved area of the spherical segment is [Wilson's Solid Geometry, page 58] $2\pi rh$, which is the first result required.

Again, if x is the radius of the circle required, its area is πx^2, which is to be equal to $2\pi rh$; hence $x^2=2rh$, or x is a mean proportional between AB and AH. But, since BPA is a right-angled triangle, and PH is perpendicular to AB, AP is a mean proportional between AB and AH, and is therefore equal to x, the radius of the circle required.

9. *Two right cones stand on a common circular base; show how to determine the radius of a sphere whose entire surface is equal to the sum of the surfaces of the cones.*

Let a, b be the slant heights of the given cones, r the radius of their common base; then their areas are respectively πra, πrb.

Let x be the radius of the sphere required; its area is therefore $4\pi x^2$, and we have

$$4\pi x^2=\pi r(a+b),$$

Hence
$$(2x)^2=r(a+b),$$

or the diameter of the sphere is a mean proportional between the radius of the base and the sum of the slant heights of the cones.

10. *Prove that the angle subtended at a focus of a conic by any chord is bisected by the line joining the focus to the pole of the chord.*

[In other words, tangents to a conic subtend equal or supplementary angles at the focus, for which easy proofs are to be found in all the books on Conic Sections.]

SECOND B.Sc. PASS EXAMINATION, 1879.

PURE MATHEMATICS.

Monday, October 27, 1879.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
Rev. Prof. TOWNSEND, M.A., F.R.S.

Morning, 10 *to* 1.

ALGEBRA AND TRIGONOMETRY.

1. *Assuming the truth of the binomial theorem for positive integers, show, on elementary principles, that it holds for fractional and negative indices also.*

[See Todhunter's Algebra, ch. xxxvi.]

2. *A system of any number of binomial factors each the sum of two squares being supposed multiplied together, show, by any method, that their product is the sum of two squares.*

Take a pair of factors of the given kind, viz.

$$a^2+b^2, \quad c^2+d^2;$$

then we have

$$(a^2+b^2)(c^2+d^2)=a^2c^2+b^2d^2+b^2c^2+a^2d^2$$
$$=(ac\pm bd)^2+(bc\mp ad)^2. \quad . \quad . \quad (1)$$

Thus the product of two such factors can be expressed in two ways as the sum of two squares: we may write them respectively A^2, B^2, where $A=ac\pm bd$, $B=bc\mp ad$.

Take now a third factor; then

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)=(A^2+B^2)(e^2+f^2)$$
$$=(Ae\pm Bf)^2+(Be\mp Af)^2, \quad . \quad (2)$$

as in (1).

Thus the product of three such factors can also be expressed as the sum of two squares; and so on for any number. We may notice that, since there are two pairs of values for A, B, and each pair gives two results in (2), we have four pairs of squares whose respective sums are equal to the product of three factors of the given form; and so on.

3. *Given that* $(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)=p^2+q^2$, *find, as an example of the preceding, the values of* p *and* q *in terms of* a, b, c, *&c.*

By the preceding result we have

$$(a^2+b^2)(c^2+d^2)=(ac\pm bd)^2+(bc\mp ad)^2$$
$$= \quad A^2 \quad + \quad B^2;$$

$$(e^2+f^2)(g^2+h^2)=(eg\pm fh)^2+(fg\mp eh)^2$$
$$= \quad C^2 \quad + \quad D^2;$$

where A, B; C, D have respectively two pairs of values.

Thus

$$(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)=(A^2+B^2)(C^2+D^2)$$
$$=(AC\pm BD)^2+(BC\mp AD)^2$$
$$=p^2+q^2.$$

It is clear that each pair of values of A, B, with each pair of values of C, D, gives us two pairs of values for p, q; thus we have in all eight solutions. One of these, namely that obtained by taking the upper sign throughout, is

$$p=AC+BD=(ac+bd)(eg+fh)+(bc-ad)(fg-eh),$$
$$q=BC-AD=(bc-ad)(eg+fh)-(ac+bd)(fg-eh).$$

4. *Assuming that*

$$\log(\cos\alpha+i\sin\alpha)+\log(\cos\beta+i\sin\beta)$$
$$+\log(\cos\gamma+i\sin\gamma)+\&c.\ldots$$

to any number of terms $=\log(\cos\phi+i\sin\phi)$ *where* $i=\sqrt{-1}$, *find, by any method, the value of* ϕ *in terms of* α, β, γ, *&c.*

This may be deduced from De Moivre's Theorem [Todhunter's Trigonometry, § 267], or may be established thus. Take n terms, viz. those involving α, β, γ, ... λ; then we have, by hypothesis,

$$\log(\cos\alpha+i\sin\alpha)+\log(\cos\beta+i\sin\beta)+\ldots$$
$$+\log(\cos\lambda+i\sin\lambda)$$
$$=\log(\cos\phi+i\sin\phi).$$

Add one more term, $\log(\cos\mu+i\sin\mu)$, to each side, then

$$\log(\cos\alpha+i\sin\alpha)+\log(\cos\beta+i\sin\beta)+\ldots$$
$$+\log(\cos\mu+i\sin\mu)$$
$$=\log(\cos\phi+i\sin\phi)+\log(\cos\mu+i\sin\mu)$$
$$=\log\{(\cos\phi+i\sin\phi)(\cos\mu+i\sin\mu)\}$$
$$=\log\{\cos(\phi+\mu)+i\sin(\phi+\mu)\}.$$

But, by hypothesis, the left hand side of this identity is equal to $\log(\cos\psi+i\sin\psi)$; thus

$$\log(\cos\psi+i\sin\psi)=\log\{\cos(\phi+\mu)+i\sin(\phi+\mu)\};$$

whence

$$\psi=\phi+\mu;$$

or adding a term $\log(\cos\mu+i\sin\mu)$ to the sum of n similar terms has the effect of increasing ϕ by μ.

But one term is $\log(\cos\alpha+i\sin\alpha)$; hence the sum of two terms is $\log\{\cos(\alpha+\beta)+i\sin(\alpha+\beta)\}$; and so on, till we have

$$\phi=\alpha+\beta+\ldots+\lambda,$$

and the sum of the n given terms is

$$\log\{\cos(\alpha+\beta+\ldots+\lambda)+i\sin(\alpha+\beta+\ldots+\lambda)\}.$$

5. *Clear of radicals the two algebraical equations*

$$x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}=0,$$
$$x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0.$$

Taking the first equation,

$$x^{\frac{1}{2}}+y^{\frac{1}{2}}=-z^{\frac{1}{2}};$$

squaring, $x+y+2x^{\frac{1}{2}}y^{\frac{1}{2}}=z,$

or $2x^{\frac{1}{2}}y^{\frac{1}{2}}=z-x-y;$

$\therefore$ $4xy=(z-x-y)^2,$

or $x^2+y^2+z^2=2xy+2yz+2zx.$

Taking the second equation

$$x^{\frac{1}{3}}+y^{\frac{1}{3}}=-z^{\frac{1}{3}};$$

cubing, $x+y+3x^{\frac{1}{3}}y^{\frac{1}{3}}(x^{\frac{1}{3}}+y^{\frac{1}{3}})=-z,$

or $x+y+z=3x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}};$

$\therefore$ $(x+y+z)^3=27xyz,$

or $x^3+y^3+z^3+3x^2y+3y^2z+3z^2x$
$+3xy^2+3yz^2+3zx^2=21xyz.$

6. *Find the values of x, y, z which satisfy the three equations*

$$x^2-yz=a^2, \quad y^2-zx=b^2, \quad z^2-xy=c^2.$$

Since $x^2-yz=a^2,$
$y^2-zx=b^2,$

subtracting, $(x-y)(x+y+z)=a^2-b^2;$

similarly $(y-z)(x+y+z)=b^2-c^2,$
$(z-x)(x+y+z)=c^2-a^2.$

Hence

$$\frac{x-y}{a^2-b^2}=\frac{y-z}{b^2-c^2}=\frac{z-x}{c^2-a^2}=\frac{1}{x+y+z}=k. \quad . \quad . \quad (1)$$

But adding the given equations, and doubling the result,

$$(x-y)^2+(y-z)^2+(z-x)^2=2(a^2+b^2+c^2);$$

whence, substituting from (1), we get

$$k^2(a^4+b^4+c^4-b^2c^2-c^2a^2-a^2b^2)=a^2+b^2+c^2.$$

But we have from the equations (1),

$$y-z=(b^2-c^2)k, \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

$$z-x=(c^2-a^2)k, \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

$$x-y=(a^2-b^2)k, \quad . \quad . \quad . \quad . \quad . \quad (4)$$

$$x+y+z=\frac{1}{k}\ ; \quad . \quad . \quad . \quad . \quad . \quad (5)$$

whence, adding (4) and (5) and subtracting (3), we find

$$3x=k(2a^2-c^2-b^2)+\frac{1}{k},$$

which, on substituting the known value of k, gives

$$x=\pm\frac{a^4-b^2c^2}{\sqrt{(a^2+b^2+c^2)(a^4+\ \ldots\ -b^2c^2-\ \ldots\ )}}$$

$$=\pm\frac{a^4-b^2c^2}{\sqrt{a^6+b^6+c^6-3a^2b^2c^2}}.$$

Similar expressions are found for y and z, where the sign for the surd must be taken throughout that was taken in the value of k.

The equation may also be solved thus:—Multiplying the original equations respectively by y, z, x, and adding, we have

$$0=a^2y+b^2z+c^2x;$$

similarly

$$0=a^2z+b^2x+c^2y;$$

whence

$$\frac{x}{a^4-b^2c^2}=\frac{y}{b^4-c^2a^2}=\frac{z}{c^4-a^2b^2}=l;$$

and substituting in any of the given equations

$$l=\pm\sqrt{a^6+b^6+c^6-3a^2b^2c^2};$$

whence the same result follows as has been found already.

7. *Given that three variables, x, y, z, are connected with a fourth, ϕ, by the three relations*

$$x=a\cos(\phi+\alpha), \quad y=b\cos(\phi+\beta), \quad z=c\cos(\phi+\gamma),$$

show that they are connected with each other by the relation

$$bc\sin(\beta-\gamma).x+ca\sin(\gamma-\alpha).y+ab\sin(\alpha-\beta).z=0.$$

Writing the equations at length,

$$\frac{x}{a}=\cos\phi\cos\alpha-\sin\phi\sin a,$$

$$\frac{y}{b}=\cos\phi\cos\beta-\sin\phi\sin\beta,$$

$$\frac{z}{c}=\cos\phi\cos\gamma-\sin\phi\sin\gamma.$$

To obtain a homogeneous result, we must eliminate $\cos\phi$ and $\sin\phi$ from the right hand, as from three linear equations in two unknowns, for which the method is the same in every case. For example, to eliminate x and y from the equations

$$p=lx+my,$$
$$p'=l'x+m'y,$$
$$p''=l''x+m''y,$$

we multiply the first equation by $l'm''-l''m'$, the second by $l''m-lm''$, the third by $lm'-l'm$, and add; our result is

$$p(l'm''-l''m')+p'(l''m-lm'')+p''(lm'-l'm)=0.$$

In the given example we have $l=\cos\alpha$, $m=-\sin\alpha$, &c.; and therefore

$$\frac{x}{a}(\sin\beta\cos\gamma-\cos\beta\sin\gamma)+\text{two similar terms}=0;$$

or

$$\frac{x}{a}\sin(\beta-\gamma)+\frac{y}{b}\sin(\gamma-\alpha)+\frac{z}{c}\sin(\alpha-\beta)=0,$$

which agrees with the given result.

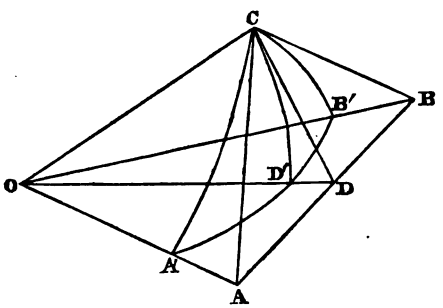
8. *Assuming the ordinary formulæ for the cosines of the angles of a spherical triangle in terms of the sides, deduce from them, in terms of the same, that for the ratio of the sine of each angle to that of the opposite side.*

[Todhunter's Spherical Trigonometry, §§ 40, 41.]

9. *Assuming that, in a right-angled plane triangle, $cp=ab$; where p is the perpendicular from the right angle on the hypothenuse c; state and prove the corresponding relation for a right-angled spherical triangle.*

Let CA′B′ be any right-angled spherical triangle in which CD′ is drawn from C at right angles to the hypo-

Fig. 14.



thenuse A′B′; also let $A'C=b'$, $A'B'=c'$, $B'C=a'$, $CD'=p'$; then shall

$$\sin p' . \sin c' = \sin a' . \sin b'.$$

At C draw a tangent plane to the sphere cutting the radii OA′, OB′, OD′ produced in A, B, D; join CA, CB, CD. Then the plane OCD′ is clearly perpendicular to the plane OAB; and the plane CAB is perpendicular to the line OC, and therefore to the plane OCD′. Hence, since the planes OAB and CAB are both perpendicular to the plane OCD′, AB (their intersection) is also perpendicular to this plane, and therefore to the lines CD, OD. Also

$$\angle ACB = \angle A'CB' = \text{a right angle} = \angle OCA$$
$$= \angle OCB = \angle OCD.$$

Now $$a = BC = OB \sin a',$$

and $$b = CA = OA \sin b';$$

$$\therefore \quad ab = OA . OB \sin a' \sin b'.$$

Again, $p = CD = OD \sin p'$,

$c = AB$;

$\therefore\ cp = AB \cdot OD \sin p'$.

But $ab = cp$;

$\therefore\ OA \cdot OB \sin a' \sin b' = AB \cdot OD \sin p'$,

and $OA \cdot OB \sin c' =$ twice the area of OAB

$= AB \cdot OD$;

$\therefore\ \sin a' \sin b' = \sin c' \sin p'$.

10. *Assuming that, in any plane triangle,*

$$\tan \tfrac{1}{2}(A+B) : \tan \tfrac{1}{2}(A-B) :: a+b : a-b,$$

with similar formulæ for the remaining pairs of angles and opposite sides; state and prove the corresponding relations for any spherical triangle.

The corresponding formula in Spherical Trigonometry is

$$\tan \tfrac{1}{2}(A+B) : \tan \tfrac{1}{2}(A-B) :: \tan \tfrac{1}{2}(a+b) : \tan \tfrac{1}{2}(a-b).$$

The deduction of this from the result in a plane triangle is tedious; and the ordinary method is therefore best, for which see Todhunter's Spherical Trigonometry, § 52.

11. *Assuming that the area of a sphere of any radius is four times that of a circle of equal radius; state and prove the ordinary formula for the area of a spherical triangle in terms of the radius of the sphere, and the spherical excess of the triangle.*

[Todhunter's Spherical Trigonometry, § 97.]

12. *The three sides a, b, c of a spherical triangle being supposed given; find in terms of them the radii of the inscribed and of the three exscribed circles of the triangle.*

[Todhunter's Spherical Trigonometry, ch. vii.]

Afternoon, 3 *to* 6.

PURE AND COORDINATE GEOMETRY.

1. *The equation in rectangular coordinates of two right lines through the origin being* $ax^2+2hxy+by^2=0$, *investigate by any method, that of the two bisectors, external and internal, of the angle they determine.*

Let the angles which the given lines make with the axis of x be θ, ϕ; then these lines are represented by the equation

$$(y-x\tan\theta)(y-x\tan\phi)=0.$$

But their equation is also

$$ax^2+2hxy+by^2=0.$$

Comparing coefficients,

$$\tan\theta+\tan\phi=-\frac{2h}{a},$$

$$\tan\theta\tan\phi=\frac{b}{a}.$$

Now let x, y be a point on one of the bisectors; then

$$y=x\tan\tfrac{1}{2}(\theta+\phi)=x\tan\alpha \text{ say,}$$

and $$\tan(\theta+\phi)=\tan 2\alpha;$$

$$\therefore \quad \frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{2\tan\alpha}{1-\tan^2\alpha},$$

or $$\frac{2h}{b-a}=\frac{2xy}{x^2-y^2}.$$

Hence $(x^2-y^2)2h+2xy(b-a)=0$,

or $$hx^2+xy(a-b)-hy^2=0,$$

represents the bisectors required.

2. *The equations, in rectangular or oblique coordinates, of two pairs of right lines through the origin being*

$$a_1x^2+2h_1xy+b_1y^2=0, \text{ and } a_2x^2+2h_2xy+b_2y^2=0,$$

investigate, by any method, the condition that the two angles they determine should divide each other harmonically.

[See Salmon's Conic Sections, § 332.]

Any line meeting the four given straight lines must be divided harmonically by them; and this condition is both necessary and sufficient. Take the line $x=c$, meeting the first pair of lines at y_1, y_2, the second at y_3, y_4; then, since the angles formed by the lines divide each other harmonically, we have

$$(y_3-y_1)(y_4-y_2)=(y_2-y_3)(y_4-y_1),$$

or

$$\begin{aligned}2y_1y_2+2y_3y_4&=y_1y_4+y_2y_3+y_2y_4+y_1y_3\\&=(y_1+y_2)(y_3+y_4). \quad . \quad . \quad . \quad . \quad (1)\end{aligned}$$

But y_1, y_2; y_3, y_4 are respectively roots of

$$a_1c^2+2h_1cy+b_1y^2=0,$$
$$a_2c^2+2h_2cy+b_2y^2=0.$$

Hence

$$y_1+y_2=-\frac{2h_1}{b_1}, \quad y_3+y_4=-\frac{2h_2}{b_2},$$

$$y_1y_2=\frac{a_1}{b_1}, \quad y_3y_4=\frac{a_2}{b_2}$$

and the condition becomes, on substituting in (1),

$$\frac{a_1}{b_1}+\frac{a_2}{b_2}=\frac{2h_1h_2}{b_1b_2},$$

or

$$a_1b_2+a_2b_1=2h_1h_2.$$

3. *The general equation of the second degree in rectangular or oblique coordinates, viz.*

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

being supposed to represent two right lines, investigate, by any method, the condition that it should, and the coordinates of their intersection.

Let (X, Y) be the required coordinates of their intersection. Then, when the origin is changed to (X, Y), we

must have a homogeneous equation of the second degree in x, y; for it must represent two lines through the origin.

Effecting the transformation by substituting

$$x+\mathrm{X},\ y+\mathrm{Y} \text{ for } x, y,$$

we have

$$ax^2+2hxy+by^2+2x(a\mathrm{X}+h\mathrm{Y}+g)+2y(h\mathrm{X}+b\mathrm{Y}+f)$$
$$+(a\mathrm{X}^2+b\mathrm{Y}^2+c+2f\mathrm{Y}+2g\mathrm{X}+2h\mathrm{XY})=0.$$

This is to be homogeneous and of the second degree; therefore

$$a\mathrm{X}+h\mathrm{Y}+g=0, \quad \ldots\ldots \quad (1)$$
$$h\mathrm{X}+b\mathrm{Y}+f=0, \quad \ldots\ldots \quad (2)$$
$$(a\mathrm{X}^2+b\mathrm{Y}^2+c+2f\mathrm{Y}+2g\mathrm{X}+2h\mathrm{XY})=0. \quad \ldots \quad (3)$$

The equations (1) and (2) give X, Y; while (3) is

$$\mathrm{X}(a\mathrm{X}+h\mathrm{Y}+g)+\mathrm{Y}(h\mathrm{X}+b\mathrm{Y}+f)+(g\mathrm{X}+f\mathrm{Y}+c)=0;$$

or, from (1) and (2),

$$g\mathrm{X}+f\mathrm{Y}+c=0; \quad \ldots\ldots \quad (4)$$

whence, eliminating X and Y from (1), (2), and (4), we have the required condition,

$$abc+2fgh-af^2-bg^2-ch^2=0;$$

or, in determinant form,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0.$$

4. *The equations, in rectangular coordinates, of three circles in a common plane being supposed given in the forms*

$$(x-a_1)^2+(y-b_1)^2=r_1^2,$$
$$(x-a_2)^2+(y-b_2)^2=r_2^2,$$
$$(x-a_3)^2+(y-b_3)^2=r_3^2,$$

determine, in the same form, that of the circle in the plane, which intersects the three at right angles.

Let the equation required be

$$(x-a)^2+(y-b)^2=r^2. \quad . \quad . \quad . \quad . \quad . \quad . \quad (A)$$

Then, since it cuts each given circle orthogonally, the square on the distance between the centres is equal to the sum of the squares on the radii.

Therefore

$$(a-a_1)^2+(b-b_1)^2=r^2+r_1^2;$$

$$\therefore \quad a^2+b^2-r^2-2aa_1-2bb_1+a_1^2+b_1^2+r_1^2=0. \quad . \quad (1)$$

Similarly,

$$a^2+b^2-r^2-2aa_2-2bb_2+a_2^2+b_2^2-r_2^2=0; \quad . \quad (2)$$

$$a^2+b^2-r^2-2aa_3-2bb_3+a_3^2+b_3^2-r_3^2=0. \quad . \quad (3)$$

Treating these as simultaneous equations for a, b, $a^2+b^2-r^2$, we can easily find first a and b, then $a^2+b^2-r^2$; and substituting the known values of a, b, we have r^2. This gives us our equation in the form required; but if we only desire the equation, without reference to its form, the simplest method is to write equation (A) thus:—

$$a^2+b^2-r^2-2ax-2by+x^2+y^2=0;$$

and with this and (1), (2), and (3), we can eliminate

$$a^2+b^2-r^2, \ a, \ b,$$

by a determinant, and we have

$$\begin{vmatrix} x^2+y^2, & x, & y, & 1 \\ a_1^2+b_1^2-r_1^2, & a_1, & b_1, & 1 \\ a_2^2+b_2^2-r_2^2, & a_2, & b_2, & 1 \\ a_3^2+b_3^2-r_3^2, & a_3, & b_3, & 1 \end{vmatrix} = 0,$$

the equation of the circle required.

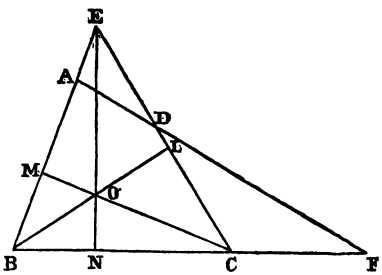
5. *Four indefinite right lines drawn arbitrarily in a plane, being supposed to determine a complete quadrilateral in the plane; show, by any method, that every two of the three diagonals of the figure divide the third harmonically.*

[Besant's Conic Sections, § 182.]

6. *Three circles, described on the three diagonals of a complete quadrilateral as diameters, being supposed all to lie in the plane of the figure, show, by any method, that the three groups of two which they determine have a common radical axis.*

Let ABCD be the quadrilateral, EF the intersections of AB, CD; BC, AD respectively. Consider the triangle BCE; let O be the intersection of BL, CM, EN the perpendiculars from B, C, E on the opposite sides; then, because AMC is a right angle, the circle on AC as diameter

Fig. 15.



will pass through M; similarly the circle on BD as diameter will pass through L, and that on EF as diameter through N. Thus MOC, LOB, EON are respectively chords of these three circles, intersecting in O. Again, since a circle will pass through B, M, L, C,

the rectangle BO . OL = the rectangle CO . OM;

and, similarly, = the rectangle NO . OE.

Thus the point O is such that the rectangles contained by the segments of chords through O to the three circles on AC, BD, EF as diameters are equal; and therefore the radical axis of each pair passes through O, the orthocentre of the triangle BCE. Similarly it passes through the orthocentres of the triangles ADE, ABF, CDF; or the three given pairs of circles have a common radical axis.

7. *Two tangents to a parabola intersect at right angles; determine, either by pure or by coordinate geometry, the locus of their intersection.*

For a proof by Pure Geometry see Besant's Geometrical Conic Sections, § 29.

We may also proceed thus:—Any two tangents to a parabola, at right angles to each other, may be written

$$y=mx+\frac{a}{m}, \quad \ldots\ldots\ldots \quad (1)$$

$$y=-\frac{x}{m}-am. \quad \ldots\ldots\ldots \quad (2)$$

By subtraction we have

$$(x+a)(m^2+1)=0.$$

Hence any pair of tangents at right angles intersect on the directrix, $x+a=0$, *except when* $m^2+1=0$.

In these two cases respectively equations (1) and (2) represent the same tangents, viz.

$$y=+(x-a)\sqrt{-1},$$
$$y=-(x-a)\sqrt{-1}.$$

Each of these is at right angles to itself; the only real point on them is the focus. Thus there are two imaginary tangents through the focus each of which is at right angles to itself; and the focus is therefore part of the locus required.

8. *Two tangents to an ellipse or hyperbola being supposed to intersect at right angles; determine, either by pure or by coordinate geometry, the locus of their intersection.*

For a proof by Pure Geometry see Besant's Geometrical Conic Sections, § 85.

Any pair of tangents mutually at right angles, to the central conic,

$$\frac{x^2}{a}+\frac{y^2}{b}=1,$$

may be written

$$y-mx=\sqrt{am^2+b},$$
$$my+x=\sqrt{a+bm^2}.$$

Squaring and adding, we have for their intersection

$$(1+m^2)(x^2+y^2)=(1+m^2)(a+b).$$

Hence $$x^2+y^2=a+b,$$

which is a circle, real in the case of the ellipse always; but in the case of the hyperbola, only when the transverse axis exceeds the conjugate.

The same remarks with reference to the exceptional case, $m^2+1=0$, apply as in question 7.

9. *A circle of any radius having its centre at the focus of a parabola being supposed to intersect the curve at two points; show, by any method, that the two lines connecting either point of intersection with the two points at which the circle intersects the transverse axis, are the tangent and normal to the curve at the point.*

Let S be the focus, P a point of intersection, LM the points of intersection of the circle with the transverse axis. Then

$\because$ SP=SL,

$\therefore$ PL is the tangent at P;

and $\because$ LPM is a right angle,

$\therefore$ PM is the normal at P.

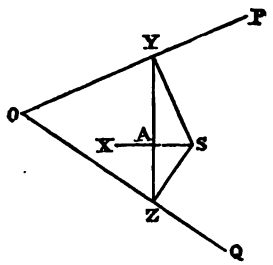
10. *A circle of any radius, passing through the foci of an ellipse or hyperbola, being supposed to intersect the curve in two points; show, by any method, that the two lines connecting either point of intersection with the two points at which the circle intersects the conjugate axis, are the tangent and normal to the curve at the point.*

Let S, H be the foci; P, Q the points of intersection of the circle with the curve, L, M with the conjugate axis. Join LP, MP. Then it is clear that LM is a diameter, and that the arcs LS, LH, MS, MH are pair and pair equal. Hence the lines LP, MP bisect respectively the internal and external angles formed by SP, HP; and are therefore the tangent and normal to the given ellipse or hyperbola at the point P.

11. *Given, of a parabola, the focus and two tangents; determine by a geometrical construction the vertex and directrix of the curve.*

Let S be the focus; OP, OQ the tangents. From S

Fig. 16.



draw SY, SZ at right angles to OP, OQ; then YZ is the tangent at the vertex. Draw SA at right angles to YZ;

then A is the vertex of the curve. . . . (1)

Produce SA to X, making AX=SA;

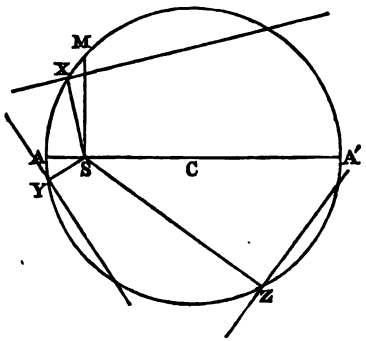
then X is the foot of the directrix. . . . (2)

12. *Given of an ellipse or hyperbola one focus and three tangents; determine by geometrical construction the centre and axes of the curve.*

Let S be the focus; from S let fall perpendiculars on the tangents, meeting them in X, Y, Z; then X, Y, Z are on the auxiliary circle. Describe a circle through X, Y, Z; it will be the auxiliary circle, its centre C will be the centre of the curve, its diameter AA′ through S the transverse axis.

If S is inside this circle, draw an ordinate, SM, at right angles to AA′: if outside, a tangent SM to the circle. In either case we have

Fig. 17.



sq. on SM = diff. of sqs. on CA, CS
= sq. on semi- conjugate axis.

Thus SM is equal in length to the semi- conjugate axis.

Tuesday, October 28, 1879.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
Rev. Prof. TOWNSEND, M.A., F.R.S.

Morning, 10 *to* 1.

DIFFERENTIAL AND INTEGRAL CALCULUS.

1. *Setting out from first principles, find the first differential coefficients with respect to the variable* x, *of the functions* $\sin x$, $\log x$, $\exp x$, *where* $\exp x = e^x$.

[See Williamson's Differential Calculus, §§ 21, 29, 30.]

NOTE.—In future we shall, when desirable, use the expression $\exp x$ instead of e^x, *e.g.* qu. 4, page 108.

2. *Starting again from first principles, state and prove the general formula for the nth differential coefficient with respect to* x, *of the product* uv *of any two functions* u *and* v *of the same variable* x.

Let $u=f(x)$, $v=\phi(x)$, then

$$\frac{d.uv}{dx}=\text{limit of } \frac{f(x+h)\phi(x+h)-f(x)\phi(x)}{h},$$

when h is indefinitely small,

$$=\text{limit of } f(x+h).\frac{\phi(x+h)-\phi(x)}{h}$$

$$+\phi(x).\frac{f(x+h)-f(x)}{h},$$

which, from definition, is

$$f(x)\frac{d\phi(x)}{dx}+\phi(x)\frac{df(x)}{dx},$$

or

$$u.\frac{dv}{dx}+v.\frac{du}{dx}.$$

Having first established this result for $\frac{d.uv}{dx}$ from first principles, we now proceed exactly as in the ordinary proof of Leibnitz's Theorem, for which see Williamson's Differential Calculus, § 48.

3. *Assuming the nth differential coefficient with respect to x of the product* $\exp(ax).\cos(bx)$ *to be of the form* $\exp(ax+nh).\cos(bx+nk)$; *find, by any method, the values of h and k in terms of a and b.*

Let $$u=\exp(ax).\cos(bx),$$

and $$\frac{d^nu}{dx^n}=\exp(ax+nh).\cos(bx+nk). \quad . \quad . \quad . \quad (1)$$

The following method for determining h, k, justifies also the given assumption.

Differentiating (1) we have

$$\frac{d^{n+1}u}{dx^{n+1}}=a\exp(ax+nh).\cos(bx+nk)$$

$$-b\exp(ax+nh).\sin(bx+nk); \quad . \quad . \quad (2)$$

but this is also, by the given assumption,

$$\exp(ax+\overline{n+1}h)\,.\cos(bx+\overline{n+1}k),$$

or $$\exp(ax+\overline{n+1}\,h)\,.\cos(bx+nk+k),$$

or $$\exp h\{\cos k\,.\exp(ax+nh)\,.\cos(bx+nk) - \sin k\,.\exp(ax+nh)\,.\sin(bx+nk)\}.$$

Equating the separate parts in this expression to the corresponding parts in (2), we have

$$\exp h\,.\cos k=a,$$

$$\exp h\,.\sin k=b;$$

or $$\tan k=\frac{b}{a},\ \exp h=\sqrt{a^2+b^2};$$

which, giving values for h, k independent of n, justifies the assumption.

4. *Assuming that $f(x)$ may, for all values of x less than unity, be expanded in an infinite series of the form*

$$a+bx+cx^2+dx^3+\&c.;$$

find, by the differential calculus, the values of the several coefficients a, b, c, d, &c.

[Williamson's Differential Calculus, § 59.]

5. *The sine of a real angle never exceeding unity in absolute value, investigate, as an example of the preceding, the development of the function $\sin^{-1}(x)$ in ascending powers of the variable x.*

Assume the expansion

$$\sin^{-1}x=a+bx+cx^2+dx^3+..$$

then $$\frac{1}{\sqrt{1-x^2}}= b+2cx+3dx^2+..$$

by differentiation; and since x^2 is less than 1, the left hand

may be expanded by the binomial theorem or the method of the preceding example. We thus obtain

$$1+\frac{1}{2}x^2+\frac{1.3}{2.4}x^4+\frac{1.3.5}{2.4.6}x^6+..=b+2cx+3dx^2+..;$$

hence $b=1,\ c=0,\ d=\frac{1}{3}.\frac{1}{2}$, &c.,

and $$\sin^{-1}x=a+x+\frac{1}{3}.\frac{1}{2}x^3+\frac{1}{5}.\frac{1.3}{2.4}x^5+..,$$

where $\sin^{-1}0=a=n\pi$, n being an even integer or zero. We must notice that the differentiation was performed on the hypothesis that $\sin^{-1}x$ lay in the first or fourth quadrant; hence the restriction in the value of n. If $\sin^{-1}x$ lies in the second or third quadrant, we have

$$\frac{d}{dx}.(\sin^{-1}x)=-\frac{1}{\sqrt{1-x^2}},$$

and hence $\sin^{-1}x=n\pi-(x+\frac{1}{3}.\frac{1}{2}x^3+$&c.), where n is an odd integer.

6. *Determine for different values of a and b, supposed both positive, the maxima and minima values of* $a\cos x+b\cos 2x$; *with the corresponding values of the variable x in the several cases.*

This problem is completely solved in Williamson's Differential Calculus, § 138: Ex. 3.

7. *A tangent to an ellipse being supposed to intersect the produced axes of the curve; find, by the differential calculus, the position and magnitude of the intercepted segment when of minimum length.*

Let ϕ be the excentric angle of the point required; then the tangent at this point to the ellipse takes the form (see Todhunter's Conic Sections, § 168)

$$\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1,$$

and the intercepted segment, r, is given by the equation

$$r^2 = a^2 \sec^2\phi + b^2 \operatorname{cosec}^2\phi;$$

$$\therefore \quad r\frac{dr}{d\phi} = a^2 \sec^2\phi \sin\phi - b^2 \operatorname{cosec}^2\phi \cos\phi.$$

But, for a minimum, $\frac{dr}{d\phi} = 0$; and hence

$$a^2 \sec^3\phi \sin\phi = b^2 \operatorname{cosec}^3\phi \cos\phi,$$

or
$$\tan\phi = \sqrt{\frac{b}{a}}.$$

Hence the coordinates x, y of the point of contact ar given by

$$x = a\cos\phi = \left(\frac{a^3}{a+b}\right)^{\frac{1}{2}},$$

$$y = b\sin\phi = \left(\frac{b^3}{a+b}\right)^{\frac{1}{2}};$$

and the segment itself is easily seen to be of length $a+b$.

The problem is most readily solved thus, without the calculus:—

$$\begin{aligned} r^2 &= a^2 \sec^2\phi + b^2 \operatorname{cosec}^2\phi \\ &= a^2 + b^2 + a^2 \tan^2\phi + b^2 \cot^2\phi \\ &= (a+b)^2 + (a\tan\phi - b\cot\phi)^2, \end{aligned}$$

whose least value is given by $\tan\phi = \sqrt{\frac{b}{a}}$, in which case $r = a+b$.

8. *Find by any method the three different values of the integral* $\int (a+b\cos x)^{-1}\,dx$ *corresponding to the three different cases of the ratio of* a *to* b *being greater than, less than, and equal to, unity.*

For the first two cases see Williamson's Integral Calculus, § 18.

In the last case, we require

$$\int (a+a\cos x)^{-1}\,dx = \int \frac{dx}{2a\cos^2\frac{x}{2}} = \frac{1}{a}\cdot\tan\frac{x}{2}.$$

9. *State the general method of integrating differentials whose coefficients are rational fractions in the variable; and apply it to the evaluation of the integral*

$$\int(1+x)^{-1}(1+x^2)^{-1}\,x\,dx$$

in finite terms.

The general method, for a full account of which see Williamson's Integral Calculus, ch. ii., consists essentially in reducing the fraction to the form (if necessary), *rational integral function of* x + *a proper fraction*, and then splitting up the proper fraction into partial fractions whose denominators are the real simple, or quadratic, factors (or, if repeated, powers of these) of the original denominator; the numerators of the partial fractions being, for each simple factor, or simple factor repeated, constant, and for each quadratic factor, or repeated quadratic factor, of the form $\mu x+\nu$. The integrals of each partial fraction can then be obtained by elementary processes.

In the example assume

$$\frac{x}{(1+x)(1+x^2)}=\frac{\lambda}{1+x}+\frac{\mu x+\nu}{1+x^2},$$

then $x=\lambda(1+x^2)+(\mu x+\nu)(1+x)$.

Putting $x=-1$ in this identity,

we have $-1=2\lambda$;

putting $x^2+1=0$,

we have $x=(\nu-\mu)+(\mu+\nu)x$.

Hence $\nu=\mu,\quad \mu+\nu=1$;

and, finally, $\lambda=-\frac{1}{2},\quad \mu=\nu=\frac{1}{2}$.

Thus

$$\int\frac{x\,dx}{(1+x)(1+x^2)}=\tfrac{1}{2}\int\left(\frac{x+1}{x^2+1}-\frac{1}{x+1}\right)dx$$

$$=\tfrac{1}{4}\log(x^2+1)-\tfrac{1}{2}\log(x+1)+\tfrac{1}{2}\tan^{-1}x.$$

10. *Apply the method of integration by parts to the reduction of the integral* $\int(a^2+x^2)^{-n}dx$, *where* n *is a positive integer; and complete the integration in finite terms for the case in which* $n=2$.

Integrating by parts, we have

$$\int(a^2+x^2)^{1-n}dx = x(a^2+x^2)^{1-n} + (n-1)\int 2x^2(a^2+x^2)^{-n}dx$$

$$= x(a^2+x^2)^{1-n} + 2(n-1)\int\{(a^2+x^2)^{1-n} - a^2(a^2+x^2)^{-n}\}dx.$$

Hence

$$2a^2(n-1)\int(a^2+x^2)^{-n}dx = (2n-3)\int(a^2+x^2)^{1-n}dx + x(a^2+x^2)^{1-n},$$

and the integral of $(a^2+x^2)^{-n}$ is reduced to depend on that of $(a^2+x^2)^{1-n}$. This process can be continued until the integration depends only on $\int(a^2+x^2)^{-1}dx$, which is known, viz.

$$\frac{1}{a}\tan^{-1}\frac{x}{a}.$$

In the given example, $n=2$, and we have

$$2a^2\int(a^2+x^2)^{-2}dx = \int(a^2+x^2)^{-1}dx + x(a^2+x^2)^{-1}$$

$$= \frac{1}{a}\tan^{-1}\frac{x}{a} + \frac{x}{a^2+x^2},$$

and the required integral is

$$\frac{1}{2a^3}\cdot\tan^{-1}\frac{x}{a} + \frac{x}{2a^2(a^2+x^2)}.$$

11. *Determine, by the integral calculus, the entire length and area of the closed curve whose equation, in rectangular coordinates, is* $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$.

The integration may either be performed in the ordinary way, or, perhaps better, by changing the variables as follows:—

Let any point on the curve be represented by the coordinates

$$x = a\cos^3\theta,$$

$$y = a\sin^3\theta;$$

then

$$\begin{aligned} ds^2 &= dx^2 + dy^2 \\ &= 9a^2(\cos^4\theta \sin^2\theta + \sin^4\theta \cos^2\theta)d\theta^2 \\ &= 9a^2 \cos^2\theta \sin^2\theta \, d\theta^2. \end{aligned}$$

Hence $ds = 3a \cos\theta \sin\theta \, d\theta.$

Now the integration for s must extend (in one quadrant) from $x=a$ to $x=0$, or from $\theta=0$ to $\theta=\frac{\pi}{2}$: and the whole curve is symmetrical as regards both axes.

Hence the length of the whole curve is

$$\begin{aligned} 4\int_0^{\frac{\pi}{2}} 3a \sin\theta \cos\theta \, d\theta &= 6a\left[\sin^2\theta\right]_0^{\frac{\pi}{2}} \\ &= 6a. \end{aligned}$$

NOTE.—Here, as elsewhere, the square brackets denote that the expression within them is taken between the limits on the right.

Again, the area A,

$$\begin{aligned} &= 4\int_0^a y \, dx \\ &= 4a^2 \int_0^{\frac{\pi}{2}} 3 \sin^4\theta \cos^2\theta \, d\theta \\ &= 3a^2 \int_0^{\frac{\pi}{2}} \sin^2\theta \sin^2 2\theta \, d\theta \\ &= \tfrac{3}{4}a^2 \int_0^{\frac{\pi}{2}} (1-\cos 2\theta)(1-\cos 4\theta) d\theta \\ &= \tfrac{3}{4}a^2 \int_0^{\frac{\pi}{2}} \left(1 - \cos 2\theta - \cos 4\theta \right. \\ &\qquad \left. + \tfrac{1}{2}(\cos 2\theta + \cos 6\theta)\right) d\theta \\ &= \frac{3\pi a^2}{8}. \end{aligned}$$

12. *Determine, by the integral calculus, the entire length and area of the closed curve whose equation, in polar coordinates, is*

$$r^{\frac{1}{2}} = a^{\frac{1}{2}} \cos \tfrac{1}{2}\theta.$$

We have here

$$r = a \cos^2 \tfrac{1}{2}\theta = \tfrac{1}{2}a(1 + \cos\theta),$$

$$\frac{dr}{d\theta} = -\tfrac{1}{2}a \sin\theta.$$

Hence

$$\frac{ds}{d\theta} = \tfrac{1}{2}\sqrt{a^2 \sin^2\theta + a^2(1+\cos\theta)^2}$$

$$= a \cos \tfrac{1}{2}\theta;$$

and

$$s = 2\int_0^{\pi} a \sin \tfrac{1}{2}\theta \, d\theta$$

$$= 4a\left[-\cos \tfrac{1}{2}\theta\right]_0^{\pi} = 4a.$$

Again,

$$\frac{dA}{d\theta} = \tfrac{1}{2}r^2 = \tfrac{1}{8}a^2(1+\cos\theta)^2;$$

hence

$$A = 2\int_0^{\pi} \tfrac{1}{8}a^2(1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \tfrac{1}{8} \cdot a^2 \int_0^{\pi} (3 + 4\cos\theta + \cos 2\theta) d\theta$$

$$= \frac{3\pi a^2}{8}.$$

Monday, October 25, 1880.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
Rev. Prof. TOWNSEND, M.A., F.R.S.

Morning, 10 *to* 1.

ALGEBRA AND TRIGONOMETRY.

1. *Solve the simultaneous equations*

$$x^2+xy+y^2=a,$$
$$x^4+x^2y^2+y^4=b.$$

The identity $x^4+x^2y^2+y^4=(x^2+xy+y^2)(x^2-xy+y^2)$ should be remembered.

Dividing the second of the given equations by the first, we find that

$$x^2-xy+y^2=\frac{b}{a},$$

and
$$x^2+xy+y^2=a.$$

Hence
$$x^2+y^2=\frac{a^2+b}{2a},$$

and
$$2xy=\frac{a^2-b}{a};$$

$$\therefore \quad (x+y)^2=\frac{3a^2-b}{2a},$$

and
$$(x-y)^2=\frac{3b-a^2}{2a};$$

whence

$$x=\tfrac{1}{2}\left[\pm\sqrt{\frac{3b-a^2}{2a}}\pm\sqrt{\frac{3a^2-b}{2a}}\right],$$

$$y=\tfrac{1}{2}\left[\mp\sqrt{\frac{3b-a^2}{2a}}\pm\sqrt{\frac{3a^2-b}{2a}}\right],$$

where the signs in x are arbitrary, but those in y are taken in the same way with those in x, *i. e.* upper with upper &c., giving four pairs of roots.

2. *Apply the Binomial Theorem to the expansion of* $\frac{1}{(1+x^2)^{\frac{1}{2}}}$ *in a series of ascending powers of* x*; and write down the coefficient of* x^{2n} *in the expansion.*

We require the expansion of $(1+x^2)^{-\frac{1}{2}}$; this is given by

$$1-\tfrac{1}{2}.x^2+\frac{-\frac{1}{2}.(-\frac{1}{2}-1)}{1.2}.x^4 + \frac{-\frac{1}{2}.(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1.2.3}.x^6+..$$

$$=1-\tfrac{1}{2}x^2+\frac{1.3}{2^2.1.2}x^4-\frac{1.3.5}{2^3.1.2.3}x^6+..,$$

where the general term is

$$(-1)^n\frac{1.3.5\ldots 2n-1}{2^n.\lfloor n}x^{2n},$$

whose coefficient may also be written

$$(-1)^n\frac{1.2.3\ldots 2n}{2^n.\lfloor n.2.4.6..2n},$$

or

$$(-1)^n\frac{\lfloor 2n}{2^{2n}(\lfloor n)^2}.$$

3. *Define a logarithm; and prove that*

$$\log\left(\frac{a}{b}\right)=\log a-\log b.$$

If

$$\log_{10}2=\cdot 3010300,$$

$$\log_{10}3=\cdot 4771213,$$

find the logarithms of

$$\sqrt[5]{6\cdot 48},\ \textit{and}\ \sqrt[4]{\cdot 0375}.$$

[Todhunter's Trigonometry, §§ 132, 136.]

$$6{\cdot}48 = 3^4 \cdot 2^3 \div 100,$$

and $$\log(6{\cdot}48) = 4\log 3 + 3\log 2 - 2$$
$$= {\cdot}8115752;$$

$$\log \sqrt[5]{6{\cdot}48} = \frac{{\cdot}8115752}{5} = {\cdot}1623150.$$

Again, $${\cdot}0375 = 3 \div (2^3 \times 10),$$

and $$\log({\cdot}0375) = \log 3 - 3\log 2 - 1 = \bar{2}{\cdot}5740313;$$

$$\therefore \quad \log \sqrt[4]{{\cdot}0375} = \tfrac{1}{4}(\bar{2}{\cdot}5740313) = \tfrac{1}{4}(-4 + 2{\cdot}5740313)$$
$$= \bar{1}{\cdot}6435078.$$

4. *Express* $\sin(x+iy)$ *and* $\cos(x+iy)$ *in the form* $X+iY$, *where* i *represents* $\sqrt{-1}$.

Assuming the exponential values of the sine and cosine of an angle (see Note, page 97) viz.

$$2\cos\theta = \exp i\theta + \exp(-i\theta),$$
$$2i\sin\theta = \exp i\theta - \exp(-i\theta),$$

we have at once

$$\sin(x+iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \tfrac{1}{2}\sin x[\exp y + \exp(-y)]$$
$$+ \tfrac{1}{2} i \cos x[\exp y - \exp(-y)], \quad . \quad . \quad (1)$$

and $$\cos(x+iy) = \cos x \cos iy - \sin x \sin iy$$
$$= \tfrac{1}{2}\cos x[\exp y + \exp(-y)]$$
$$- \tfrac{1}{2} i \sin x[\exp y - \exp(-y)]. \quad . \quad 2)$$

Thus the expressions (1), (2) are of the form required. The above results are sometimes written thus :—

$$\sin(x+iy) = \sin x \cosh y + i \cos x \sinh y,$$
$$\cos(x+iy) = \cos x \cosh y - i \sin x \sinh y.$$

5. *Eliminate ϕ between the equations*

$$x = a\sec(\phi+\alpha),$$
$$y = b\tan(\phi+\beta).$$

Let $\phi+\beta=\psi$, then $\phi+\alpha=\psi+\alpha-\beta$, and we have

$$x\cos(\psi+\alpha-\beta)=a, \quad \ldots\ldots \quad (1)$$
$$b\tan\psi \quad\quad =y. \quad \ldots\ldots \quad (2)$$

On expanding, (1) becomes

$$x\cos\psi[\cos(\alpha-\beta)-\tan\psi\sin(\alpha-\beta)]=a,$$

or $$x[\cos(\alpha-\beta)-\tan\psi\sin(\alpha-\beta)]=a\sec\psi;$$

or, by (2), and squaring,

$$x^2[b\cos(\alpha-\beta)-y\sin(\alpha-\beta)]^2=a^2b^2\sec^2\psi$$
$$=a^2b^2(1+\tan^2\psi)$$
$$=a^2(b^2+y^2);$$

$\therefore$ the result of the elimination required is

$$x^2[b\cos(\alpha-\beta)-y\sin(\alpha-\beta)]^2=a^2(b^2+y^2).$$

6. *Find the value of x which satisfies the equation*

$$\sin^{-1}\left(\frac{x}{a}\right)+\sin^{-1}\left(\frac{x}{b}\right)+\sin^{-1}\left(\frac{x}{c}\right)=\pi.$$

We have $$\sin^{-1}\frac{x}{a}+\sin^{-1}\frac{x}{b}=\pi-\sin^{-1}\frac{x}{c};$$

$$\therefore \quad \sin\left(\sin^{-1}\frac{x}{a}+\sin^{-1}\frac{x}{b}\right)=\sin\left(\pi-\sin^{-1}\frac{x}{c}\right),$$

or $$\frac{x}{a}\sqrt{1-\frac{x^2}{b^2}}+\frac{x}{b}\sqrt{1-\frac{x^2}{a^2}}=\frac{x}{c},$$

whence $x=0$;

or $$\sqrt{b^2-x^2}+\sqrt{a^2-x^2}=\frac{ab}{c},$$

and $$\therefore \quad \sqrt{b^2-x^2}-\sqrt{a^2-x^2}=\frac{c(b^2-a^2)}{ab}$$

[see question 8, p. 43];

$$\therefore \quad 2\sqrt{b^2-x^2}=\frac{a^2b^2+c^2b^2-c^2a^2}{abc},$$

or
$$4(b^2-x^2)=\frac{(a^2b^2+b^2c^2-c^2a^2)^2}{a^2b^2c^2},$$

or
$$4x^2=\frac{-a^4b^4-b^4c^4-c^4a^4+2a^4b^2c^2+2b^4c^2a^2+2c^4a^2b^2}{a^2b^2c^2},$$

or
$$x=\pm\tfrac{1}{2}\frac{\sqrt{(bc+ca+ab)(-bc+ca+ab)(bc-ca+ab)(bc+ca-ab)}}{abc}.$$

7. *In a spherical triangle prove the formula*

$$\cos c=\cos a\cos b+\sin a\sin b\cos C,$$

and deduce the corresponding formula by aid of the polar triangle.

[See Todhunter's Spherical Trigonometry, §§ 37, 47, 60.]

8. *What theorems respecting a plane triangle are deduced from the preceding when we regard the radius of the sphere as becoming infinitely great?*

From the first formula we deduce [see Todhunter's Spherical Trigonometry, § 132] the theorem

$$2ab\cos C=a^2+b^2-c^2.$$

From the formula

$$\cos C=-\cos A\cos B+\sin A\sin B\cos c$$

we find, on making the radius of the sphere infinite, while the sides of the triangle remain finite, that $\cos c$ ultimately becomes unity, and thus, in a plane triangle,

$$\cos C=-\cos A\cos B+\sin A\sin B.$$

or
$$\cos C=-\cos(A+B),$$

or
$$A+B+C=180^\circ,$$

the theorem required.

9. *In a spherical triangle, right-angled at* C, *prove the formula* $\tan c \cos A = \tan b$.

[See Todhunter's Spherical Trigonometry, § 62.]

10. *In a spherical triangle prove that the great circles drawn through the angular points at right angles to the opposite sides pass through a common point.*

[See Todhunter's Spherical Trigonometry, §§ 188, 189.]

Afternoon, 3 to 6.

PURE AND COORDINATE GEOMETRY.

1. *Find the area of the triangle formed by the lines joining the points* (ab), (a_1b_1), (a_2b_2), *referred to oblique coordinates.*

[See Todhunter's Conic Sections, § 11.]

The result is

$$\tfrac{1}{2}\{ab_1 - a_1b + a_1b_2 - a_2b_1 + a_2b - ab_2\} \sin\omega,$$

ω being the angle between the axes.

2. *Find the equation to the right line which passes through the intersection of* $ax+by+c=0$ *and* $a'x+b'y+c'=0$; *and also through the intersection of* $lx+my+n=0$ *and* $l'x+m'y+n'=0$.

The equation

$$\lambda(ax+by+c)+\lambda'(a'x+b'y+c')=0, \quad . \quad . \quad (A)$$

being of the first degree, represents a straight line; also since the coordinates of the point common to the first pair of given straight lines satisfy it, it represents a straight line through their intersection; but in order that it may represent the line passing also through the intersection of the last pair of lines, we must have, on substituting the coordinates of this point,

$$\lambda\{a(mn'-m'n)+b(nl'-n'l)+c(lm'-l'm)\}$$
$$+\lambda'\{a'(mn'-m'n)+b'(nl'-n'l)+c'(lm'-l'm)\}=0.$$

This gives us the ratio of λ to λ'; and substituting in A we have the equation to the line required, viz.

$$[a'(mn'-m'n)+..+..](ax+by+c)$$
$$=[a(mn'-m'n)+..+..](a'x+b'y+c'),$$

or, in determinant form,

$$\begin{vmatrix} a' & b' & c' \\ l & m & n \\ l' & m' & n' \end{vmatrix} (ax+by+c)$$
$$= \begin{vmatrix} a & b & c \\ l & m & n \\ l' & m' & n' \end{vmatrix} (a'x+b'y+c').$$

3. *Prove that the locus of a point from which the lengths of the tangents to two circles are in a constant ratio is a circle; and show that the three circles have a common radical axis.*

Let the given circles be

$$(x-a)^2+(y-b)^2=c^2,$$
$$(x-h)^2+(y-k)^2=r^2.$$

Also let (ξ, η) be a point on the locus required; then the squares of the tangents from (ξ, η) to the given circles are respectively

$$(\xi-a)^2+(\eta-b)^2-c^2,$$

and

$$(\xi-h)^2+(\eta-k)^2-r^2;$$

hence, since these are in a fixed ratio, say $l^2 : m^2$, we have

$$\frac{(\xi-a)^2+(\eta-b)^2-c^2}{l^2} = \frac{(\xi-h)^2+(\eta-k)^2-r^2}{m^2},$$

the equation to the locus required.

It represents a circle, except when $l=m$, in which case it becomes a straight line. Moreover, if the equations to the given circles be written $S_1=0$, $S_2=0$, the equation of this third circle becomes $m^2S_1-l^2S_2=0$; and it therefore

passes through the intersections, real or imaginary, of S_1, S_2; and therefore has a common chord with them; and therefore all three have a common radical axis.

4. *Find the equation to the circle circumscribing the triangle formed by the lines* $x\cos\alpha+y\sin\alpha-p=0$, $x\cos\alpha'+y\sin\alpha'-p'=0$, and $x\cos\alpha''+y\sin\alpha''-p''=0$.

The curve

$$\lambda(x\cos\alpha'+y\sin\alpha'-p')(x\cos\alpha''+y\sin\alpha''-p'')$$
$$+\mu(x\cos\alpha''+y\sin\alpha''-p'')(x\cos\alpha+y\sin\alpha-p)$$
$$+\nu(x\cos\alpha+y\sin\alpha-p)(x\cos\alpha'+y\sin\alpha'-p')=0$$

is satisfied by the three intersections of the given lines; and hence, if we can make it a circle, it will be the circle required.

Equating, then, the coefficients of x^2 and y^2, and making that of xy vanish, we obtain

$$\lambda\cos(\alpha'+\alpha'')+\mu\cos(\alpha''+\alpha)+\nu\cos(\alpha+\alpha')=0,$$
$$\lambda\sin(\alpha'+\alpha'')+\mu\sin(\alpha''+\alpha)+\nu\sin(\alpha+\alpha')=0;$$

whence

$$\frac{\lambda}{\sin(\alpha'-\alpha'')}=\frac{\mu}{\sin(\alpha''-\alpha)}=\frac{\nu}{\sin(\alpha-\alpha')};$$

and therefore

$$\sin(\alpha'-\alpha'')(x\cos\alpha'+y\sin\alpha'-p')(x\cos\alpha''+y\sin\alpha''-p'')$$
$$+..+..=0$$

is the equation required.

5. *Prove that the tangents drawn to a parabola from any point on the directrix intersect at right angles; and that the chord joining their points of contact passes through the focus.*

[See Besant's Conic Sections § 29; Drew's Conic Sections, § 8.]

6. *Prove that the sum of the squares on a pair of conjugate diameters of an ellipse is constant.*

[See Besant's Conic Sections, § 79; Drew's Conic Sections, § 37.]

7. *Show how to determine the area of an ellipse, and construct a circle whose area shall be equal to the sum of the areas of two given ellipses.*

[See Besant's Conic Sections, § 155; Drew's Conic Sections, § 42.]

The area of an ellipse being a mean proportional between the areas of the circles on the major and minor axes as diameters, it must be equal to that of the circle described on the mean proportional between the major and minor axes as diameters.

Let AB, BC be the mean proportionals between the axes of the two given ellipses, and let them be placed at right angles with one extremity of each coincident. Join AC. Then, by Euclid, any rectilinear figure on AC=sum of similar and similarly described figures on BC, AB; but the circles on AC, BC, AB as diameters are limiting cases of similar rectilinear figures.

Hence

Circle on CA as diameter

=circle on BC as diameter

+circle on AB as diameter

=sum of the areas of the two given ellipses;

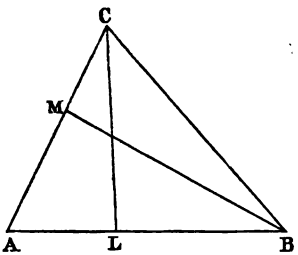
for each ellipse is equal to one of the circles on AB, BC.

8. *In a given triangle* ABC *the point* L *is taken on* AB, *and* M *on* AC, *so that* AL : LB=CM : AM. *Find the equation to the locus of the points of intersection of the lines* CL, BM.

Take AB, AC as axes; let AB=a, AC=b, AL=la,

AM$=mb$. Then, from the given proportion we readily deduce $l+m=1$.

Fig. 18.



Now, the equations to CL, BM are at once seen to be

$$\left.\begin{aligned}\frac{x}{la}+\frac{y}{b}&=1,\\ \frac{x}{a}+\frac{y}{mb}&=1,\end{aligned}\right\}, \text{ where } l+m=1.$$

Hence $$l=\frac{bx}{a(b-y)}, \quad m=\frac{ay}{b(a-x)},$$

and the equation to the intersection of CL, BM is therefore

$$\frac{bx}{a(b-y)}+\frac{ay}{b(a-x)}=1;$$

or $$x^2b^2+y^2a^2+xyab-2ab^2x-2ba^2y+a^2b^2=0,$$

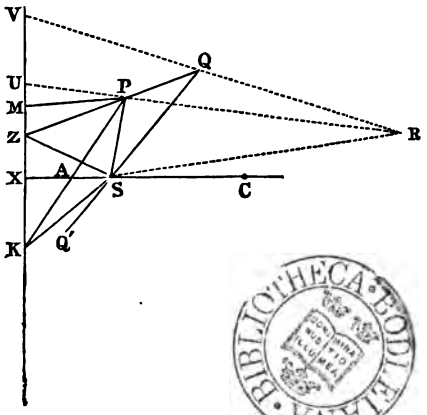
or $$\frac{x}{a}+\frac{y}{b}-1=\sqrt{\frac{xy}{ab}},$$

which represents an ellipse touching AB, AC at B and C.

9. *Being given a focus and three points on the ellipse, determine by a geometrical construction the centre and axes of the curve.*

Let S be the focus; P, Q, R the given points. Join SP, SQ, produce QS to Q′, and bisect the angle PSQ′ by

Fig. 19.



SZ, meeting QP produced in Z. Then [Besant's Conics, § 5] Z is on the directrix.

Similarly we get two other points U, V on the directrix; and thus the directrix ZU is known.

Draw SX perpendicular to ZU, and bisect the exterior angle to PSX, by the line SK meeting the directrix in K; then, if PK cuts SX in A, A is the vertex.

For, drawing PM perpendicular to ZU,

$\because$ KS bisects the exterior angle to $\angle$PSX,

$\therefore$ PS : SA :: PK : AK :: PM : AX;

$\therefore$ PS : PM :: SA : AX;

$\therefore$ A is the vertex.

Then divide AS produced in C so that CS : CA :: AS : AX; then C will be the centre, CA the major axis, and the minor axis will be readily found by constructing a right-angled triangle whose hypothenuse is CA and height equal to CS; the base will be equal to CB.

10. *If any right line intersect an hyperbola, prove that the portions intercepted between the curve and its asymptotes are equal.*

[See Besant's Conic Sections, § 116; Drew's Conic Sections, § 58.]

Tuesday, October 26, 1880.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
Rev. Prof. TOWNSEND, M.A., F.R.S.

Morning, 10 *to* 1.

DIFFERENTIAL AND INTEGRAL CALCULUS.

1. *Explain what is meant by the differential coefficient of any expression, and find the differential coefficients of*

$$\log(\tan x), \quad \frac{1-x}{\sqrt{1+x^2}}, \quad \text{and} \quad \log(\sqrt{x+a}+\sqrt{x+b}).$$

[See Williamson's Differential Calculus, § 7; Todhunter's Differential Calculus, § 24.]

The differential coefficients required may be easily worked from first principles; but the ordinary method is the shortest, thus:—

$$\frac{d\log\tan x}{dx}=\frac{1}{\tan x}\cdot\frac{d\tan x}{dx}=\frac{\sec^2 x}{\tan x}=\frac{1}{\sin x\cos x};$$

$$\frac{d}{dx}\left(\frac{1-x}{\sqrt{1+x^2}}\right)=\frac{-1\,.\,\sqrt{1+x^2}-\frac{x}{\sqrt{1+x^2}}(1-x)}{1+x^2}=-\frac{1+x}{(1+x^2)^{\frac{3}{2}}}.$$

$$\frac{d}{dx}\log(\sqrt{x+a}+\sqrt{x+b})$$

$$=\frac{1}{\sqrt{x+a}+\sqrt{x+b}}\cdot\frac{d}{dx}(\sqrt{x+a}+\sqrt{x+b})$$

$$=\tfrac{1}{2}\cdot\frac{\frac{1}{\sqrt{x+a}}+\frac{1}{\sqrt{x+b}}}{\sqrt{x+a}+\sqrt{x+b}}=\frac{1}{2\sqrt{x+a}\sqrt{x+b}}.$$

2. *If* $u=(\sin x)^n$, *find the value of* $\frac{d^4u}{dx^4}$.

Here

$$u=(\sin x)^n;$$

$$\therefore \frac{du}{dx}=n\cos x(\sin x)^{n-1},$$

and $$\frac{d^2u}{dx^2}=n(n-1)\cos^2x(\sin x)^{n-2}-n(\sin x)^n$$

$$=n(n-1)(\sin x)^{n-2}-n^2(\sin x)^n. \quad . \quad . \quad . \quad . \quad (1)$$

Hence, differentiating (1) twice, we have

$$\frac{d^4u}{dx^4}=n(n-1)\frac{d^2}{dx^2}(\sin x)^{n-2}-n^2\frac{d^2}{dx^2}(\sin x)^n;$$

but, writing $(n-2)$ for n in (1),

$$\frac{d^2}{dx^2}(\sin x)^{n-2}=(n-2)(n-3)(\sin x)^{n-4}-(n-2)^2(\sin x)^{n-2};$$

$$\therefore \frac{d^4u}{dx^4}=n(n-1)[(n-2)(n-3)(\sin x)^{n-4}-(n-2)^2(\sin x)^{n-2}]$$

$$-n^2[n(n-1)(\sin x)^{n-2}-n^2(\sin x)^n]$$

$$=n(n-1)(n-2)(n-3)(\sin x)^{n-4}$$

$$-2n(n-1)(n^2-2n+2)(\sin x)^{n-2}+n^4(\sin x)^n.$$

3. *Assuming that* $f(x)$ *can be expanded in a series of the form*

$$a_0+a_1x+a_2x^2+\text{ \&c.},$$

show how the values of a_0, a_1, a_2 *&c. can be determined by the differential calculus; and apply the method to the expansions of* $(1+x)^n$ *and* a^x.

[See question 4, p. 99.]

$f(x)=(1+x)^n$, $\quad\therefore f(0)=1,$

$f'(x)=n(1+x)^{n-1}$, $\quad\therefore f'(0)=n,$

$f''(x)=n(n-1)(1+x)^{n-2}$, $\quad\therefore f''(0)=n(n-1)$, &c.;

$$f^r(x)=n(n-1)\ldots(n-r+1)(1+x)^{n-r},$$

$$\therefore\quad f^r(0)=n(n-1)\ldots(n-r+1).$$

Hence

$$(1+x)^n=1+nx+\frac{n(n-1)}{1\,.\,2}x^2+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3}x^3+\ldots$$

This series terminates at the $(n+1)$th term, if n is a positive integer; otherwise it has an infinite number of terms.

For the second example see Williamson's Differential Calculus, § 60.

4. *Explain the method for finding the maxima and minima values of a function of x, and show how they are distinguished from each other.*

[See Todhunter's Differential Calculus, § 211; Williamson's Differential Calculus, § 138.]

5. *Find the maxima and minima values of* $\frac{x+a}{x^2+a^2}$, *and of* $\frac{a}{\tan x}+\frac{b}{\tan 2x}$.

In the first case we have

$$y=\frac{x+a}{x^2+a^2},$$

$$\therefore\quad \frac{dy}{dx}=\frac{a^2-2ax-x^2}{(x^2+a^2)^2},$$

and

$$\frac{d^2y}{dx^2}=\frac{2x^3+6ax^2-6a^2x-2a^3}{(x^2+a^2)^3}.$$

$\frac{dy}{dx}$ vanishes when $x=a(-1\pm\sqrt{2})$.

Now, since y is never infinite, and is positive when x is greater than $-a$, we infer that the positive value of x makes y a maximum, and the negative value makes y a minimum.

This may also be shown by examining the sign of the second differential coefficient when $x=a(-1\pm\sqrt{2})$.

To find the two values of y, we have

$$y=\frac{x+a}{x^2+a^2}=\frac{-1\pm\sqrt{2}+1}{a(3+1\mp2\sqrt{2})}=\frac{-1}{2a(1\mp\sqrt{2})}=\frac{1\pm\sqrt{2}}{2a},$$

where the positive value is a maximum, and the negative a minimum.

This might have been established by Algebra alone, thus :—

Solving for y,

$$\begin{aligned}2xy&=1\pm\sqrt{1+4ay-4a^2y^2}\\&=1\pm\sqrt{(1+\alpha y)(1-\beta y)},\end{aligned}$$

where clearly the greatest value of y is $\frac{1}{\beta}$, and the least $-\frac{1}{\alpha}$; but α, β are respectively $2a(1+\sqrt{2})$ and $2a(\sqrt{2}-1)$: hence our result.

In the second example we have

$$\begin{aligned}y &= a\cot x+b\cot 2x,\\ \therefore\ \frac{dy}{dx}&=-a\operatorname{cosec}^2x-2b\operatorname{cosec}^2 2x\\&=-\tfrac{1}{2}\operatorname{cosec}^2x(2a+b\sec^2x).\end{aligned}$$

Now, if a and b are of the same sign, or if $b>2a$ numerically, this expression can never change its sign; there is therefore no maximum or minimum value of y in these cases.

If b is less than or equal to $2a$, numerically, and of opposite sign, $\frac{dy}{dx}$ will vanish when $\cos x=-\frac{b}{2a}$; and in this case y is a maximum or minimum, according to the quadrant in which x lies, and the sign of a. The values of y are readily found to be $\pm\sqrt{-b(2a+b)}$.

The whole problem may be solved far more simply by Algebra and Trigonometry, as follows :—

$$y=a\cot x+b\cot 2x.$$

Solving for $\tan x$, we have

$$b \tan x = -y \pm \sqrt{y^2 + 2ab + b^2}.$$

Thus y may have any value from $-\infty$ to $+\infty$ (since $\tan x$ may have any value) if $2ab + b^2$ is positive, *i. e.* if b and $2a + b$ are of the same sign, which agrees with our previous condition. If, however, $b(2a + b)$ is negative, the maximum and minimum values of y are respectively

$$+\sqrt{-b(2a+b)} \text{ and } -\sqrt{-b(2a+b)}.$$

6. *Integrate the expressions*

$$\tan^{-1} x dx\,; \quad \frac{dx}{(1+x^2)\sqrt{1-x^2}}.$$

Integrating the first expression by parts, we have

$$\int \tan^{-1} x dx = x \tan^{-1} x - \int \frac{x dx}{1+x^2}$$
$$= x \tan^{-1} x - \tfrac{1}{2} \log (1+x^2).$$

In the second expression, put $x = \sin\theta$; then we have $dx = \cos\theta\, d\theta$, and

$$\int \frac{dx}{(1+x^2)\sqrt{1-x^2}} = \int \frac{d\theta \sec^2\theta}{1+2\tan^2\theta} = \int \frac{d\tan\theta}{1+2\tan^2\theta}$$
$$= \frac{\tan^{-1}(\tan\theta\sqrt{2})}{\sqrt{2}} = \frac{\tan^{-1}\left(\dfrac{x\sqrt{2}}{\sqrt{1-x^2}}\right)}{\sqrt{2}}.$$

7. *Find the values of the definite integrals*

$$\int_\beta^\alpha \frac{dx}{\sqrt{(x-\beta)(\alpha-x)}},$$

$$\int_0^h x(x-1)(x-2)(x-3)dx.$$

In the first integral let $x=y+\frac{1}{2}(\alpha+\beta)$, and call $\alpha-\beta$, $2c$; the integral becomes

$$\int_{-c}^{c}\frac{dy}{\sqrt{(y+c)(c-y)}}=\int_{-c}^{c}\frac{dy}{\sqrt{c^2-y^2}}$$
$$=\left[\sin^{-1}\frac{y}{c}\right]_{-c}^{c}=\pi.$$

Again, the second integral is

$$\int_0^h (x^4-6x^3+11x^2-6x)dx$$
$$=\frac{6h^5-45h^4+110h^3-90h^2}{30}.$$

8. *Show how to integrate* $\frac{f(x)}{\phi(x)}\,dx$, *where* $f(x)$ *and* $\phi(x)$ *are rational algebraic functions of* x, *pointing out the different forms of integrals that may take place.*

[Williamson's Integral Calculus, Ch. II.]

9. *Find the area and length of the cardioid*

$$r=a(1+\cos\theta).$$

This question, by writing the curve in the form

$$r=2a\cos^2\frac{\theta}{2},$$

is at once seen to be similar to quest. 12, page 105, the only difference being that we have here $2a$ instead of a.

Hence the length becomes $8a$, and the area $\frac{3\pi a^2}{2}$.

10. *The coordinates of any point on an epitrochoid are represented by the equations*

$$x=a\cos\theta+b\cos m\theta,$$
$$y=a\sin\theta+b\sin m\theta;$$

prove that the length of any arc on the curve is equal to that of an ellipse, and find the lengths of the axes of the ellipse.

[For a similar problem see Williamson's Integral Calculus, § 164.]

In the given curve

$$dx = -(a \sin\theta + mb \sin m\theta)d\theta,$$

$$dy = \quad (a \cos\theta + mb \cos m\theta)d\theta;$$

$$\therefore (ds)^2 = (dx)^2 + (dy)^2 = [a^2 + 2amb\cos(1-m)\theta + m^2b^2](d\theta)^2.$$

Thus the arc from $\theta = \alpha$ to $\theta = \beta$ is given by

$$s = \int_\alpha^\beta (a^2 + 2amb\cos(1-m)\theta + m^2b^2)^{\frac{1}{2}}\, d\theta.$$

Now, let $(1-m)\theta = 2\phi$; and we have

$$s = \int (a^2 + 2amb\cos 2\phi + m^2b^2)^{\frac{1}{2}}\, d\phi \div \tfrac{1}{2}(m-1),$$

the limits being $\frac{1}{2}(1-m)\beta$, $\frac{1}{2}(1-m)\alpha$.

This may be written

$$s = \int (p^2 \sin^2\phi + q^2\cos^2\phi)^{\frac{1}{2}}\, d\phi, \quad \ldots \quad (1)$$

with the same limits as before, where

$$p(m-1) = 2(a-mb),$$

$$q(m-1) = 2(a+mb);$$

but (1) is also the expression for the arc of an ellipse whose axes are p, q, the coordinates of any point on which may be represented by $x = p\cos\phi$, $y = q\sin\phi$.

Monday, October 24, 1881.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
Rev. Prof. TOWNSEND, M.A., F.R.S.

Morning, 10 *to* 1.

ALGEBRA AND TRIGONOMETRY.

1. *Find the value of* x *which satisfies the equation*

$$\sqrt{(x^2-a^2)(x^2-b^2)}+\sqrt{(x^2-b^2)(x^2-c^2)}+\sqrt{(x^2-c^2)(x^2-a^2)}+x^2=0,$$

and exhibit the result in a symmetric form.

We have

$$\sqrt{(x^2-a^2)(x^2-b^2)}+\sqrt{(x^2-c^2)(x^2-a^2)}=-\sqrt{(x^2-b^2)(x^2-c^2)}-x^2;$$

squaring,

$$(x^2-a^2)(x^2-b^2)+(x^2-c^2)(x^2-a^2)+2(x^2-a^2)\sqrt{(x^2-b^2)(x^2-c^2)}=(x^2-b^2)(x^2-c^2)+x^4+2x^2\sqrt{(x^2-b^2)(x^2-c^2)},$$

or

$$(a^2b^2+a^2c^2-b^2c^2)-2x^2a^2=2a^2\sqrt{(x^2-b^2)(x^2-c^2)};$$

squaring again, and simplifying, we have

$$\begin{aligned}4x^2a^2b^2c^2&=4a^4b^2c^2-(a^2b^2+a^2c^2-b^2c^2)^2\\&=\{a^2(b+c)^2-b^2c^2\}\{b^2c^2-a^2(b-c)^2\}\\&=\{a(b+c)-bc\}\{a(b+c)+bc\}\\&\quad\{bc+a(b-c)\}\{bc-a(b-c)\}\\&=16s(s-bc)(s-ca)(s-ab),\end{aligned}$$

where $2s=bc+ca+ab.$

Hence $$x=\pm\frac{2\sqrt{s(s-bc)(s-ca)(s-ab)}}{abc}.$$

In equations of this kind the final result does not always satisfy the given equation; for instance, in this example, every term as written will be found to be real and therefore positive, the various radicals cannot then be all taken with positive signs. The examination of this point is tedious and uninteresting; one result alone may be mentioned. If any two of the three quantities bc, ca, ab are together greater than the third, all three radicals must be taken with a negative sign.

Compare, also, question 6, page 109.

2. *Determine the expansion of* $(x+y)^n$ *when* n *is a fraction, and show that the series can always be exhibited in a convergent form.*

If x is less than y, the expansion is made thus:

$$(x+y)^n = y^n\left(1+\frac{x}{y}\right)^n$$

$$= y^n\left(1+n\frac{x}{y}+\frac{n(n-1)}{1.2}\left(\frac{x}{y}\right)^2+..\right),$$

which is always convergent, since x is less than y.

Similarly, if y is less than x, we have

$$(x+y)^n = x^n\left(1+\frac{y}{x}\right)^n$$

$$= x^n\left(1+n\frac{y}{x}+\frac{n(n-1)}{1.2}\left(\frac{y}{x}\right)^2+..\right),$$

which is also convergent.

If $x=y$, the series becomes $(2x)^n$.

3. *Prove that* $\dfrac{(a+ib)^m}{(a'+ib')^n}$ *is reducible to the form*

$$\rho\,(\cos\theta+i\sin\theta),$$

where $i=\sqrt{-1}$, *and find the values of* ρ *and* θ.

Any imaginary quantity, as for instance $a+ib$, can be reduced to the form $r(\cos A+i\sin A)$, where r and A are

real. For, equating real and imaginary parts in the assumed equality $a+ib=r(\cos A+i\sin A)$, we have at once

$$a=r\cos A, \quad b=r\sin A;$$

thus
$$\frac{b}{a}=\tan A, \quad \ldots\ldots\ldots \quad (1)$$

and
$$b^2+a^2=r^2; \quad \ldots\ldots\ldots \quad (2)$$

these last two equations always give real values for A and r.

Hence
$$\begin{aligned}(a+ib)^m&=\{r(\cos A+i\sin A)\}^m\\&=r^m(\cos A+i\sin A)^m\\&=r^m(\cos mA+i\sin mA).\end{aligned}$$

Similarly $(a'+ib')^n=r'^n(\cos nA'+i\sin nA')$;

where
$$r'=\sqrt{a'^2+b'^2}, \quad \ldots\ldots\ldots \quad (3)$$

$$\tan A'=\frac{b'}{a'}. \quad \ldots\ldots\ldots \quad (4)$$

Thus
$$\begin{aligned}\frac{(a+ib)^m}{(a'+ib')^n}&=\frac{r^m(\cos mA+i\sin mA)}{r'^n(\cos nA'+i\sin nA')}\\&=r^mr'^{-n}(\cos mA+i\sin mA)\\&\qquad(\cos nA'-i\sin nA')\\&=r^mr'^{-n}\{\cos(mA-nA')\\&\qquad+i\sin(mA-nA')\},\end{aligned}$$

which is of the required form; and

$$\left.\begin{aligned}\rho&=r^m\div r'^n,\\ \theta&=mA-nA';\end{aligned}\right\}$$ where r, r', A, A' are given by (1), (2), (3), (4).

4. *Find the sum of n terms of the series*

$$1+3x+5x^2+7x^3+\&c.=0.$$

Let S be the sum required, we have

$$S=1+3x+5x^2+..+(2n-1)x^{n-1},$$

and
$$xS= \quad x+3x^2+..+(2n-3)x^{n-1}+(2n-1)x^n;$$

thus $$S(1-x)=1+2x+2x^2\ldots+2x^{n-1}-(2n-1)x^n$$
$$=1+\frac{2(x-x^n)}{1-x}-(2n-1)x^n$$
$$=\frac{1+x-(2n+1)x^n+(2n-1)x^{n+1}}{1-x};$$

and $$S=\frac{1+x-(2n+1)x^n+(2n-1)x^{n+1}}{(1-x)^2}.$$

5. *Resolve*

$$\frac{9x^2-7x+2}{x^4-x^3+x-1}$$

into partial fractions.

Here
$$x^4-x^3+x-1=(x-1)(x+1)(x^2-x+1).$$

Assume then
$$\frac{9x^2-7x+2}{x^4-x^3+x-1}=\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2-x+1};$$

then
$$9x^2-7x+2=A(x+1)(x^2-x+1)+B(x-1)(x^2-x+1)$$
$$+(Cx+D)(x^2-1).$$

In this identity

put $x=1$, thus $A=2$,

put $x=-1$, $B=-3$;

and, lastly, equate the coefficients of x^3, and the constant terms, thus:—

$$0=A+B+C,$$
$$2=A-B-D.$$

Hence $$C=1,\quad D=3,$$

and $$\frac{9x^2-7x+2}{x^4-x^3+x-1}=\frac{2}{x-1}-\frac{3}{x+1}+\frac{x+3}{x^2-x+1}.$$

The given expression is therefore resolved into real partial fractions, the last of which could itself be readily resolved into two partial fractions with imaginary denominators of the first degree, viz. $x-\frac{1}{2}(1\pm\sqrt{-3})$.

6. *Solve the equation*

$$\tan^{-1}\frac{x}{a}+\tan^{-1}\frac{x}{b}+\tan^{-1}\frac{x}{c}+\tan^{-1}\frac{x}{d}=\frac{\pi}{2}.$$

Let $\tan^{-1}\frac{x}{a}=A$, with similar expressions for the other inverse functions; then

$$A+B+C+D=\frac{\pi}{2},$$

$$A+B=\frac{\pi}{2}-(C+D),$$

and $$\tan(A+B)=\cot(C+D),$$

or $$\tan(A+B)\tan(C+D)=1,$$

$$\frac{\tan A+\tan B}{1-\tan A\tan B}\cdot\frac{\tan C+\tan D}{1-\tan C\tan D}=1.$$

Expanding and reducing,

$$1-(\tan A\tan B+\tan A\tan C+\ \ldots\ \&c.)$$
$$+\tan A\tan B\tan C\tan D=0;$$

substituting in terms of $\frac{x}{a}$ &c.,

$$\frac{x^4}{abcd}-x^2\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{ad}+\frac{1}{bc}+\frac{1}{bd}+\frac{1}{cd}\right)+1=0,$$

or $$x^4-x^2(bc+ca+ab+ad+bd+cd)+abcd=0:$$

thus

$$x=\tfrac{1}{2}[bc+ca+ab+ad+bd+cd\pm\sqrt{(bc+ca+\ldots+cd)^2-4abcd}];$$

whence we can find x, which has four values, of which, if a negative one be taken, we must take one of the inverse functions as positive, *i. e.* greater than a right angle, or the given equation will not hold.

7. *Find the sum of the series*

$$\sin\theta-\tfrac{1}{2}\sin 2\theta+\tfrac{1}{3}\sin 3\theta-\ \ldots$$

ad infinitum.

If $x = \cos\theta + i\sin\theta$,

where $i = \sqrt{-1}$, we know that

$$2i\sin\theta = x - \frac{1}{x},$$

$$2i\sin 2\theta = x^2 - \frac{1}{x^2},$$

. . . . . .

$$2i\sin n\theta = x^n - \frac{1}{x^n}.$$

The given series is therefore

$$\frac{1}{2i}\left\{\left(x-\frac{1}{x}\right)-\frac{1}{2}\left(x^2-\frac{1}{x^2}\right)+\frac{1}{3}\left(x^3-\frac{1}{x^3}\right)-\ldots\right\}$$

$$=\frac{1}{2i}\left\{\left(x-\frac{1}{2}x^2+\frac{1}{3}x^3-\ldots\right)-\left(\frac{1}{x}-\frac{1}{2}\frac{1}{x^2}+\frac{1}{3}\frac{1}{x^3}-\ldots\right)\right\}$$

$$=\frac{1}{2i}\left\{\log(1+x)-\log\left(1+\frac{1}{x}\right)\right\}$$

$$=\frac{1}{2i}\left(\log\cdot\frac{1+x}{1+\frac{1}{x}}\right)=\frac{1}{2i}\log x.$$

But we know that $x = e^{\theta i}$;

hence $\log x = \theta i$;

and the sum of the series is therefore $\frac{\theta}{2}$.

8. A, B, C *are three given points on the surface of a sphere. Find the locus of a fourth point* P *when* $\cos AP + \cos BP = \cos CP$.

Let $PC = \theta$, $\angle PCB = \phi$; then θ, ϕ are the angular co-ordinates of P referred to C as origin, and CB as initial line. Then, with the usual notation, for the triangle ABC we have, from the triangle PCB,

$$\cos BP = \cos\theta\cos a + \sin\theta\sin a\cos\phi;$$

and from the triangle PCA

$$\cos \mathrm{AP} = \cos\theta \cos b + \sin\theta \sin b \cos(c-\phi).$$

But $$\cos \mathrm{AP} + \cos \mathrm{BP} = \cos \mathrm{CP} = \cos\theta;$$

therefore

$$\cos\theta\{\cos a + \cos b - 1\} + \sin\theta\{\sin a \cos\phi + \sin b \cos(c-\phi)\} = 0,$$

which is the equation to a great circle.

See Todhunter's Spherical Trigonometry, §§ 133, 168–170.

9. *In a spherical triangle prove the formula*

$$\tan\frac{\mathrm{A}}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}},$$

and deduce the corresponding formula by aid of the polar triangle.

[Todhunter's Spherical Trigonometry, §§ 45, 50.]

10. *Show how to solve a spherical triangle when two sides and the included angle are given.*

[Todhunter's Spherical Trigonometry, § 82.]

Afternoon, 3 *to* 6.

PURE AND COORDINATE GEOMETRY.

1. *Find the value of h so that the equation*

$$x^2 + hxy - 6y^2 - 7x - y + 12 = 0$$

shall represent a pair of lines.

The equation

$$ax^2 + by^2 + c + 2fy + 2gx + 2hxy = 0$$

represents two straight lines if

$$af^2 + bg^2 + ch^2 - abc - 2fgh = 0.$$

In the given example this reduces, on clearing of fractions, to

$$2 \cdot 1^2 - 12 \cdot 7^2 + 24h^2 + 2 \cdot 12 \cdot 24 - 2 \cdot 1 \cdot 7 \cdot h = 0,$$

or
$$12h^2 - 7h - 5 = 0;$$

hence
$$h = 1, \quad \text{or} \quad -\frac{5}{12}.$$

The pairs of lines represented are respectively

$$(x + 3y - 4)\,(x - 2y - 3) = 0,$$

and
$$(4x + 9y - 12)(3x - 8y - 12) = 0.$$

For another method, see Salmon's Conic Sections, ex. 1, § 77*.

2. *Find the equation of the circle which passes through the points* (1, 6), (3, 2), *and* (5, 4).

Let the required equation be

$$x^2 + y^2 + ax + by + c = 0;$$

then, substituting successively the coordinates of the given points, we have

$$1 + 36 + a + 6b + c = 0,$$
$$9 + 4 + 3a + 2b + c = 0,$$
$$25 + 16 + 5a + 4b + c = 0;$$

whence we readily obtain

$$a = -\frac{16}{3}, \quad b = -\frac{26}{3}, \quad c = \frac{61}{3};$$

and the curve is

$$x^2 - \frac{16}{3}x + y^2 - \frac{26}{3}y + \frac{61}{3} = 0,$$

or
$$\left(x - \frac{8}{3}\right)^2 + \left(y - \frac{13}{3}\right)^2 = \frac{50}{9}.$$

3. *Find the locus of the centre of a conic which passes through four given points.*

This problem is solved by various methods in Salmon's Conic Sections, ex. 3. § 151, ex. 15. § 328. It may also be treated thus:—Let the four given points be, in areal coordinates, $l\alpha = \pm m\beta = \pm n\gamma$; then, if the conic

$$p\alpha^2+q\beta^2+r\gamma^2+2s\beta\gamma+2t\gamma\alpha+2v\alpha\beta=0$$

passes through them, we find at once that

$$s=t=v=0, \quad \text{and} \quad \frac{p}{l^2}+\frac{q}{m^2}+\frac{r}{n^2}=0. \quad . \quad . \quad . \quad . \quad (1)$$

Now the pole of the line at infinity, $\alpha+\beta+\gamma=0$, with respect to the conic

$$p\alpha^2+q\beta^2+r\gamma^2=0,$$

is the point

$$p\alpha=q\beta=r\gamma.$$

But this is the centre of the conic; and, substituting in (1) we see that its locus is

$$\frac{1}{\alpha l^2}+\frac{1}{\beta m^2}+\frac{1}{\gamma n^2}=0.$$

4. *Given of a parabola its focus, any point on the curve, and a tangent to the curve, find the position of its vertex.*

Let LM be the given tangent, S the focus, P the given point. From S draw SY perpendicular to LM, and produce SY to N, making NY = YS; then N is the foot of the perpendicular from the point of contact of LM on the directrix.

Describe a circle with centre S and radius SP, and from N draw a tangent to this circle: this is the directrix. Then draw SX perpendicular to the directrix, bisect SX in A. A is clearly the vertex of the curve.

We may notice that the directrix is obtained from its common tangency to two envelopes; for, given the focus S and the tangent LM, the directrix passes through the point N, obtained as shown above; and given the focus S and the point P, the directrix touches a circle with centre

P, radius PS. Hence, given focus, tangent, and point, the directrix must fulfil both the above conditions, or it must be a tangent from N to the circle described. We have, in general, two positions of the directrix and two parabolas.

5. *Prove that any triangle may be orthogonally projected into an equilateral triangle.*

Let ABC be any triangle, and D, E, F the middle points of its sides; G the point where the lines AD, BE, CF meet. Then an ellipse may be described with G as centre, and passing through A, B, C. Now project this ellipse into a circle, namely that on the minor axis, and let the projections of A, B, C, &c., be A′, B′, C′ . . . &c.

Then,

$\because$ the diameter AGD bisects the chord BC in the ellipse,

$\therefore$ „ A′GD′ „ „ B′C′ in the circle;

$\therefore$ A′GD′ is also at right angles to B′C′;

$\therefore$ $A'B' = A'C'$;

similarly

$$B'A' = B'C'.$$

$\therefore$ the triangle A′B′C′ is equilateral, and it is the projection of the triangle ABC.

Note. As regards the construction of the ellipse (which is the least that can be described about the triangle ABC), it may be noticed that GA, BC being conjugate, we can readily find the length also of the diameter conjugate to GA; for if GK be this diameter, we have

$$GK^2 : GA^2 :: BD^2 : GA^2 - GD^2;$$

but

$$GA = 2GD,$$

$$\therefore \quad 3GK^2 = 4BD^2 = BC^2.$$

Now, having two conjugate diameters GA, GK in position and magnitude, we may find the axes of the ellipse by the ordinary construction, for which see Besant's Conic Sections, §§ 216, 217.

6. *If the equation* $ax^2+2hxy+by^2+2gx+2fy+c=0$ *be transformed from one system of rectangular axes to another, prove that the quantities* $a+b$ *and* $ab-h^2$ *remain unaltered by the transformation.*

What does the general equation represent when $ab-h^2=0$, *and what when* $a+b=0$?

A change of origin does not alter the coefficients a, b, h, properties of which we have to establish; and in any change of axes the new coefficients of x^2, xy, y^2 will only depend on the angle of transformation and the original coefficients.

Suppose then, that, by any change of axes without change of origin the expression $ax^2+2hxy+by^2$ becomes $a'x'^2+2h'x'y'+b'y'^2$, both systems being rectangular, we have also $x^2+y^2=x'^2+y'^2$; and by combining these two,

$$ax^2+2hxy+by^2+\lambda(x^2+y^2)$$
$$=a'x'^2+2h'x'y'+b'y'^2+\lambda(x'^2+y'^2),$$

where λ is any constant.

Now the expression on the left has always two factors which must repectively reduce, on transformation by the ordinary formulæ

$$x=x'\cos\theta-y'\sin\theta,$$
$$y=x'\sin\theta+y'\cos\theta,$$

to those of the expression on the right.

Thus, if *we so choose* λ *that the factors on the left are equal, those on the right will also be equal*; or, in other words, the same numerical values of λ make both the expressions

$$ax^2+2hxy+by^2+\lambda(x^2+y^2),$$
$$a'x'^2+2h'x'y'+by'^2+\lambda(x'^2+y'^2),$$

perfect squares.

The equations giving these values of λ are seen to be

$$(a+\lambda)(b+\lambda)-h^2=0,$$
$$(a'+\lambda)(b'+\lambda)-h'^2=0;$$

or

$$\lambda^2+\lambda(a+b)+ab-h^2=0,$$
$$\lambda^2+\lambda(a'+b')+a'b'-h'^2=0.$$

These equations, being the same, give us

$$a+b=a'+b', \quad ab-h^2=a'b'-h'^2;$$

or the quantities to be considered, viz. $a+b$, $ab-h^2$, are not altered by *any* change of rectangular axes.

Suppose that we so change our axes that h' vanishes, as is clearly possible; then we have the equation

$$ax^2+2hxy+by^2+..=0$$

reduced to $$a'x'^2+b'y'^2+\ldots\ldots=0, \quad \ldots\ldots \quad (1)$$

where $$a'+b'=a+b, \quad a'b'=ab-h^2.$$

Thus, if $ab-h^2=0$, a' or b' must vanish, and (1) represents a parabola. If $a+b=0$, we have $a'=-b'$, and (1) represents a rectangular hyperbola.

7. In an ellipse, if focal chords be drawn parallel to a pair of conjugate diameters, prove that their sum is constant.

Let CP, CD be two conjugate semidiameters, and QSq, RSr the parallel focal chords; A, A′ the vertices of the conic, SL the semilatus rectum. Then

rect. QS . Sq : rect. AS . SA′ :: sq. on CP : sq. on AC.

But SL is the harmonic mean between SQ and Sq; and therefore

$$\text{rect. } Qq \text{ . SL} = 2 \text{ rect. SQ . S}q.$$

Or, remembering that

$$\text{rect. AC . SL} = \text{sq. on BC} = \text{rect. AS . SA}',$$

$$\text{rect. QS . S}q : \text{rect. AS . SA}' :: Qq : 2\text{AC};$$

$$\therefore \quad Qq : 2\text{AC} :: \text{sq. on CP} : \text{sq. on AC, from above,}$$

or $$\text{rect. } Qq \text{ . AC} = 2 \text{ sq. on CP.}$$

Similarly $$\text{rect. R}r \text{ . AC} = 2 \text{ sq. on CD};$$

$$\therefore \quad \text{rect. } (Qq+\text{R}r)\text{AC} = 2(\text{sq. on CP} + \text{sq. on CD})$$
$$= 2(\text{sq. on AC} + \text{sq. on BC});$$

or $$Qq+\text{R}r \text{ is constant.}$$

If AC$=a$, BC$=b$, then $Qq+\text{R}r=\dfrac{2(a^2+b^2)}{a}$.

8. *Prove that the three points of intersection of the opposite sides of a hexagon inscribed in a conic lie in a right line.*

[Pascal's Theorem. See Todhunter's Conic Sections, § 342; Salmon's Conic Sections, §§ 267, 354, ex. xi.]

9. *Prove that the radius of curvature at any point in an ellipse is* $\frac{b'^3}{ab}$, *where* b' *is the semidiameter conjugate to that drawn to the point.*

It is easily proved (see Besant's Conic Sections, § 162) that the radius of curvature at a point P on an ellipse is $\frac{CD^2}{PF}$, where PF is the perpendicular from P to CD, the diameter conjugate to CP; then, since

$$PF \cdot CD = AC \cdot BC = ab,$$

we have

$$\frac{CD^3}{PF} = \frac{CD^3}{PF \cdot CD} = \frac{b'^3}{ab},$$

the required expression for the radius of curvature.

10. *A sphere is inscribed in a right cone. If any tangent plane be drawn to the sphere, prove that the point of contact is a focus of the section made in the cone.*

This is the ordinary method by which it is proved that any plane section of a right cone is one of the curves generally defined by a different property, and called conic sections.

See Wilson's Solid Geometry and Conic Sections, book v.; Besant's Conic Sections, ch. vi.; Drew's Conic Sections, ch. iv.

Tuesday, October 25, 1881.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
Rev. Prof. TOWNSEND, M.A., F.R.S.

Morning, 10 *to* 1.

DIFFERENTIAL AND INTEGRAL CALCULUS.

1. *Differentiate the expressions*

$$\tan^{-1}\frac{x}{\sqrt{1-x^2}}, \quad \log\sqrt{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}-x}}, \quad \textit{and}$$

$$x\sec^{-1}x-\tfrac{1}{2}\log(1+x^2).$$

If, in the first expression, $\sin\theta=x$, then

$$\tan\theta=\frac{x}{\sqrt{1-x^2}}, \quad \text{and} \quad \tan^{-1}\frac{x}{\sqrt{1-x^2}}=\theta=\sin^{-1}x.$$

Thus the differential coefficient sought is that of

$$\sin^{-1}x, \quad \text{or} \quad \frac{1}{\sqrt{1-x^2}}.$$

Again, the second differential coefficient is

$$\tfrac{1}{2}\left[\frac{d\log(\sqrt{1+x^2}+x)}{dx}-\frac{d\log(\sqrt{1+x^2}-x)}{dx}\right],$$

$$=\tfrac{1}{2}\left(\frac{1}{\sqrt{1+x^2}}+\frac{1}{\sqrt{1+x^2}}\right)=\frac{1}{\sqrt{1+x^2}}.$$

This equality between the differential coefficients of $\sqrt{1+x^2}+x$ and $\sqrt{1+x^2}-x$ is explained by the identity

$$(\sqrt{1+x^2}+x)(\sqrt{1+x^2}-x)=1+x^2-x^2=1;$$

thus

$$\log(\sqrt{1+x^2}+x)+\log(\sqrt{1+x^2}-x)=0.$$

Lastly

$$\frac{d}{dx} \cdot x \sec^{-1} x = \sec^{-1} x + \frac{1}{\sqrt{x^2-1}},$$

and

$$\frac{d}{dx} \cdot \tfrac{1}{2} \log(1+x^2) = \frac{x}{1+x^2}.$$

2. *If* $y = x^5 \log x$, *find* $\frac{d^6 y}{dx^6}$.

We have

$$y = x^5 \log x,$$

$$\frac{dy}{dx} = 5x^4 \log x + x^4.$$

Now it is clear that in the sixth differentiation of y the fifth differential coefficient of x^4 will vanish; and the same result will be true of all the terms which appear except the first, so that finally

$$\frac{d^6 y}{dx^6} = \frac{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{x} = \frac{120}{x}.$$

In general, if

$$y = x^{n-1} \log x,$$

then

$$\frac{d^n y}{dx^n} = \frac{\lfloor n-1}{x}.$$

See Williamson's Differential Calculus, § 42.

3. *Prove Maclaurin's Theorem, and apply it to the expansion of* $\sin x$ *and* $\sin^{-1} x$.

Williamson's Differential Calculus, §§ 59, 61; see also question 5, page 99.

4. *Find the maximum or minimum values of*

$$a^2 \cos^n x + b^2 \sec^n x, \quad 2 \cos x + 3 \cos 2x, \quad \text{and} \quad \frac{x^2}{(x+2)(x+3)}.$$

In the first expression let $\cos x = y$; then we require the maximum or minimum values of u where

$$u = a^2 y^n + b^2 y^{-n}.$$

Differentiating with regard to x,

$$n(a^2y^{n-1}-b^2y^{-n-1})\frac{dy}{dx}=0,$$

$$\left.\begin{aligned} y^n=\pm\frac{b}{a}=\cos^n x \\ \text{or} \qquad \sin x=0 \end{aligned}\right\}.$$

Several cases may arise. First, let b be greater than or equal to a; then the only admissible values of x are given by $\sin x=0$, $\cos x=\sec x=\pm 1$, $x=0$, π, 2π, 3π, &c. In this case the corresponding values of u are a^2+b^2, $\pm(a^2+b^2)$, a^2+b^2, $\pm(a^2+b^2)$, ... where, when n is even, the upper sign is taken, and all the values of u are minima; and when n is odd the lower sign, and the values are alternate minima and maxima. . . . . (I.)

If, however, b is less than a, let $\cos\alpha=\sqrt[n]{\frac{b}{a}}$, where α is acute, then $\cos x$ may be equal to ± 1 or $\pm\cos\alpha$. In the former case we have u alternately a maximum and a minimum when n is odd, and a maximum when n is even, as x passes through the values 0, π, 2π, 3π ....; and in the latter case we have, in succession, two minima and two maxima when n is odd, or minima alone when n is even, as x passes through the values $-\alpha$, α, $\pi-\alpha$, $\pi+\alpha$. . . . (II.)

These results may be obtained more simply by ordinary Algebra, thus:—

$$\begin{aligned} u&=a^2\cos^n x+b^2\sin^n x \\ &=a^2+b^2+(1-\cos^n x)(b^2\sec^n x-a^2). \end{aligned}$$

Now, if $b>a$, and n even, all the terms are positive, and the least value of u, and there is no minimum, is

$$a^2+b^2. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

If n is odd, there is also a minimum of the same value; but when $\cos x$ changes sign, u changes from $+\infty$ to $-\infty$, it may be written

$$u=-(a^2+b^2)+(1+\cos^n x)(b^2\sec^n x+a^2),$$

which is clearly a maximum when $\cos x = -1$, its value being $-(a^2+b^2)$; for the term

$$(b^2 \sec^n x + a^2) \text{ is negative.} \quad \ldots \quad (2)$$

Next, let b be $< a$; we have the same two expressions for u; but now a^2+b^2 is the maximum value, $-(a^2+b^2)$ the minimum value; for the term

$b^2 \sec x - a^2$ is negative when $\cos x = 1$,

and

$b^2 \sec^n x + a^2$ is positive when $\cos x = -1$. . . . (3)

Lastly
$$\begin{aligned} & a^2 \cos^n x + b^2 \sec^n x \\ = \; & 2ab + \sec^n x (a \cos^n x - b)^2 \\ = \; & -2ab + \sec^n x (a \cos^n x + b)^2, \end{aligned}$$

which is a minimum, viz. $2ab$, when $\cos x = \sqrt[n]{\frac{b}{a}}$, and a maximum, viz. $-2ab$, when $\cos x = -\sqrt[n]{\frac{b}{a}}$ and n is odd. . . . . (4)

These results are equivalent to those already obtained.

The maxima and minimum values of $2\cos x + 3\cos 2x$ are readily fonnd to be 5, 1, and $-3\frac{1}{8}$ respectively; this example being a particular case of question 6, page 100.

In the third example, call the given expression u; then, splitting it into partial fractions,

$$u = 1 + \frac{4}{x+2} - \frac{9}{x+3};$$

$$\therefore \quad \frac{du}{dx} = \frac{5x^2+12x}{(x+2)^2(x+3)^2},$$

which vanishes when $x=0$, or $-\frac{12}{5}$, or ∞, and is infinite when $x=-2$ or -3.

When x passes through the value 0, the derived function changes from $-$ to $+$, and thus $x=0$ gives a minimum value of u, viz. zero. . . . . (1)

Again, when x passes through the value $-\frac{12}{5}$, the derived function clearly changes from + to −, and we have a maximum value of u, viz. -24. . . . . (2)

When x is infinite the derived function remains positive; and there is therefore no maximum or minimum in this case. . . . . (3)

When x passes through the values -2, -3, the same remarks apply as in (3), and there is no maximum or minimum. . . . . (4)

5. *If all the sides of a triangle receive indefinitely small increments, determine the corresponding increments in its angles.*

It will be obviously simplest to use a formula involving one angle and the sides: such a formula is

$$a^2=b^2+c^2-2bc\cos A;$$

then
$$\begin{aligned} ada &= bdb+cdc+bc\sin A\,.\,dA-\cos A\,.\,d(bc),\\ &= db(b-c\cos A)+dc(c-b\cos A)\\ &\quad +bc\sin A\,.\,dA,\\ &= db\,.\,a\cos C+dc\,.\,a\cos B+bc\sin A\,.\,dA. \end{aligned}$$

Thus
$$dA=\frac{ada-adb\,.\cos C-adc\,.\cos B}{bc\sin A},$$

or
$$\frac{dA}{a}=\frac{da-db\,.\cos C-dc\,.\cos B}{bc\sin A},$$

with two similar expressions for $\frac{dB}{b}$ and $\frac{dC}{c}$.

6. *If* $y=a\cos(\log z)+b\sin(\log z)$, *prove that*

$$z^2\frac{d^2y}{dz^2}+z\frac{dy}{dz}+y=0.$$

Here
$$y= a\cos(\log z)+b\sin(\log z),$$

$$\therefore\ \frac{dy}{dz}=-\frac{a\sin(\log z)}{z}+\frac{b\cos(\log z)}{z},$$

or $$z\frac{dy}{dz} = -a\sin(\log z) + b\cos(\log z).$$

Differentiating,

$$z\frac{d^2y}{dz^2} + \frac{dy}{dz} = -\frac{a\cos(\log z)}{z} - \frac{b\sin(\log z)}{z} = -\frac{y}{z};$$

thus $$z^2\frac{d^2y}{dz^2} + z\frac{dy}{dz} + y = 0.$$

7. *Find the values of the following integrals :—*

$$\int \sin^4 x \cos^5 x dx, \quad \int e^{ax}\sin bx\, dx, \quad \int \frac{dx}{x^2(1+x^4)^{\frac{3}{4}}}.$$

We have

$$\int \sin^4 x \cos^5 x dx = \int \sin^4 x(1-\sin^2 x)^2 \cos x dx$$

$$= \int (\sin^4 x - 2\sin^6 x + \sin^8 x)\cos x \,.\, dx$$

$$= \frac{\sin^5 x}{5} - \frac{2\sin^7 x}{7} + \frac{\sin^9 x}{9}.$$

For the next integral see Williamson's Integral Calculus, ex. 4, § 21.

$$\int \frac{dx}{x^2(1+x^4)^{\frac{3}{4}}} = -\int \frac{y^3 dy}{(1+y^4)^{\frac{3}{4}}}, \text{ where } y = \frac{1}{x},$$

$$= -(1+y^4)^{\frac{1}{4}} = -\frac{(1+x^4)^{\frac{1}{4}}}{x}.$$

8. *Evaluate the following definite integrals :—*

$$\int_0^1 \frac{dx}{1+x+x^2}, \quad \int_0^{\frac{\pi}{2}} x^2\cos x dx, \quad \int_0^{\pi} \frac{dx}{1-2a\cos x + a^2}.$$

For the first integral,

$$\int_0^1 \frac{dx}{1+x+x^2} = \int_0^1 \frac{dx}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

$$= \left[\sqrt{\frac{4}{3}} \cdot \tan^{-1} \frac{x+\frac{1}{2}}{\sqrt{\frac{3}{4}}}\right]_0^1$$

$$= \frac{2}{\sqrt{3}}\left(\tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{2\pi}{3\sqrt{3}}.$$

For the second integral,

$$\int x^2 \cos x dx = \int x^2 \frac{d(\sin x)}{dx} \cdot dx$$

$$= x^2 \sin x - \int 2x \sin x dx, \text{ by parts,}$$

$$= x^2 \sin x + 2x \cos x - 2 \sin x, \text{ by parts;}$$

thus

$$\int_0^{\frac{\pi}{2}} x^2 \cos x dx = \left[x^2 \sin x + 2x \cos x - 2 \sin x\right]_0^{\frac{\pi}{2}} = \frac{\pi^2}{4} - 2.$$

For the third integral we have

$$\frac{1-a^2}{1-2a\cos x + a^2} = 1 + 2a \cos x + \ldots + 2a^n \cos nx + \ldots,$$

where a is less than unity; and

$$\int_0^\pi \frac{dx(1-a^2)}{1-2a\cos x+a^2} = \int_0^\pi (1 + 2a\cos x + \ldots + 2a^n \cos nx + \ldots) dx$$

$$= \left[x + 2a \sin x + 2a^2 \frac{\sin 2x}{2} + \ldots + 2a^n \frac{\sin nx}{n} + \ldots\right]_0^\pi = \pi.$$

Thus

$$\int_0^\pi \frac{dx}{1-2a\cos x + a^2} = \frac{\pi}{1-a^2}.$$

Suppose, however, that a is greater than unity; let $a=\frac{1}{b}$, we then require $\int_0^\pi \frac{b^2 dx}{1-2b\cos x+b^2}$.

This integral is $\frac{\pi b^2}{1-b^2}$, since b is less than unity, or $\frac{\pi}{a^2-1}$.

Lastly, let $a=1$, we have

$$\int_0^\pi \frac{dx}{1-2a\cos x+a^2}=\frac{1}{4}\int_0^\pi \operatorname{cosec}^2\frac{x}{2}\,dx=\infty .$$

Hence the required integral is

$\frac{\pi}{1-a^2}$ if a is less than unity,

$\frac{\pi}{a^2-1}$ if a is greater than unity,

infinite if a is equal to unity.

9. *Find the whole area of the curve*

$$\left(\frac{x}{a}\right)^{\frac{2}{5}}+\left(\frac{y}{b}\right)^{\frac{2}{5}}=1.$$

This being clearly a curve symmetrical about both axes, the whole area is four times that in the first quadrant, or $4\int_0^a y\,dx$.

Now assume $x=a\cos^5\theta$,

$y=b\sin^5\theta$;

then

$$dx=-5a\cos^4\theta\sin\theta d\theta,$$

and the limits become $0, \frac{\pi}{2}$. We require then

$$20ab\int_0^{\frac{\pi}{2}}\sin^6\theta\cos^4\theta d\theta,$$

or $$20ab\int_0^{\frac{\pi}{2}}\{\sin^6\theta-2\sin^8\theta+\sin^{10}\theta\}d\theta,$$

which, on integrating by the usual method (see Williamson's Integral Calculus, §§ 93, 94) becomes

$$20ab\cdot\frac{\pi}{2}\left\{\frac{1.3.5}{2.4.6}-2.\frac{1.3.5.7}{2.4.6.8}+\frac{1.3.5.7.9}{2.4.6.8.10}\right\},$$

or $$\frac{15\pi ab}{128}.$$

10. *Find the length of an arc of the catenary whose equation is*

$$y=\frac{a}{2}\left(e^{\frac{x}{a}}+e^{-\frac{x}{a}}\right);$$

and also of the parabola

$$y^2=2px.$$

Both these examples are worked in Williamson's Integral Calculus, §§ 151, 152.

FIRST B.A. AND B.Sc. HONOURS EXAMINATION.

Monday, August 2, 1880.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

Morning, 10 *to* 1.

ALGEBRA AND TRIGONOMETRY.

1. *If* $\frac{\lambda+\mu x}{\lambda'+\mu' x}$ *be substituted for* x *in the equation*

$$a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0;$$

find the values of the roots of the transformed equation in terms of α, β, γ, δ *the roots of the proposed.*

In the transformed equation the four values of

$$(\lambda+\mu x)\div(\lambda'+\mu' x)$$

are clearly α, β, γ, and δ; thus the values of x are given by the equations

$$\frac{\lambda+\mu x}{\lambda'+\mu' x}=\alpha, \text{ or } \beta, \text{ or } \gamma, \text{ or } \delta.$$

Hence $x=\frac{\lambda'\alpha-\lambda}{\mu-\mu'\alpha}$, or $\frac{\lambda'\beta-\lambda}{\mu-\mu'\beta}$, or $\frac{\lambda'\gamma-\lambda}{\mu-\mu'\gamma}$, or $\frac{\lambda'\delta-\lambda}{\mu-\mu'\delta}$.

2. *In the same case if the roots of the transformed equation are* 0, 1, κ, ∞ *respectively, find the value of* κ *in terms of* α, β, γ, δ.

Taking the roots in the same order as above, we have

$$\lambda'\alpha-\lambda=0, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$
$$\lambda'\beta-\lambda=\mu-\mu'\beta, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$
$$\lambda'\gamma-\lambda=\kappa(\mu-\mu'\gamma), \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$
$$\mu-\mu'\delta=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Hence from these equations we easily find

$$\kappa=\frac{(\alpha-\gamma)(\beta-\delta)}{(\alpha-\beta)(\gamma-\delta)}.$$

3. *Solve the equation*

$$px^4+(q+3ap)x^3+(r+3aq+2a^2p)x^2 +(3ar+2a^2q)x+2a^2r=0.$$

This equation is easily seen to be

$$(px^2+qx+r)(x^2+3ax+2a^2)=0.$$

Hence $x=-\frac{q\pm\sqrt{q^2-4pr}}{2p}$, $-a$, or $-2a$.

4. *Find the condition, among its coefficients, that the expression* $a_0x^4+4a_1x^3y+6a_2x^2y^2+4a_3xy^3+y^4$ *should be reducible to the sum of the fourth powers of two linear expressions in* x *and* y.

Let the two expressions be l_1x+m_1y, l_2x+m_2y.
Then we have the identity

$$a_0x^4+\ldots+a_4y^4=(l_1x+m_1y)^4+(l_2x+m_2y)^4.$$

Hence, equating coefficients,

$$\left.\begin{aligned} a_0&= l_1^4+l_2^4,\\ a_1&=l_1^3m_1+l_2^3m_2,\\ a_2&=l_1^2m_1^2+l_2^2m_2^2,\\ a_3&=l_1m_1^3+l_2m_2^3,\\ a_4&= m_1^4+m_2^4. \end{aligned}\right\}\ldots\ldots \quad (A)$$

Hence

$$\begin{aligned} a_0a_2-a_1^2&=l_1^2l_2^2\ (l_2m_1-l_1m_2)^2,\\ a_1a_3-a_2^2&=l_1l_2m_1m_2(l_2m_1-l_1m_2)^2,\\ a_2a_4-a_3^2&=m_1^2m_2^2(l_2m_1-l_1m_2)^2; \end{aligned}$$

and therefore the condition required is

$$(a_0a_2-a_1^2)(a_2a_4-a_3^2)=(a_1a_3-a_2^2)^2.$$

We may also proceed thus:—

We have, on substituting for $a_0, a_1, \ldots$ from equations (A),

$$a_0m_1m_2 - a_1(l_1m_2+l_2m_1)+a_2l_1l_2=0,$$
$$a_1m_1m_2 - a_2(l_1m_2+l_2m_1)+a_3l_1l_2=0,$$
$$a_2m_1m_2 - a_3(l_1m_2+l_2m_1)+a_4l_1l_2=0;$$

and, eliminating m_1m_2, $l_1m_2+l_2m_1$, l_1l_2 by a determinant, our condition becomes

$$\begin{vmatrix} a_0, & a_1, & a_2 \\ a_1, & a_2, & a_3 \\ a_2, & a_3, & a_4 \end{vmatrix} = 0.$$

A general theorem, of which the above is a particular case, is proved in Salmon's Higher Algebra, §§ 168, 171.

5. *If the quotient obtained by dividing* $a_0x^m+a_1x^{m-1}+a_2x^{m-2}+\ldots$ *&c. by* $b_0x^n+b_1x^{n-1}+b_2x^{n-2}+$ *&c. be represented by* $c_0x^{m-n}+c_1x^{m-n-1}+$ *&c., show how to determine the several coefficients* $c_0, c_1, c_2,$ *&c. in the form of determinants.*

We have

$$a_0x^m+a_1x^{m-1}+\ldots=(b_0x^n+b_1x^{n-1}+\ldots) \times (c_0x^{m-n}+c_1x^{m-n-1}+\ldots)$$

identically; multiplying, therefore, and equating coefficients, we have the system of equations

$$c_0b_0 = a_0,$$
$$c_0b_1+c_1b_0 = a_1,$$
$$c_0b_2+c_1b_1+c_2b_0 = a_2, \text{ \&c.}$$

We shall proceed to find c_2 by a method that will apply equally to finding c_r. Solving for c_2 from the above equations, we have

$$c_2 \begin{vmatrix} b_0, & b_1, & b_2 \\ 0, & b_0, & b_1 \\ 0, & 0, & b_0 \end{vmatrix} = \begin{vmatrix} a_2, & b_1, & b_2 \\ a_1, & b_0, & b_1 \\ a_0, & 0, & b_0 \end{vmatrix}.$$

But the left-hand determinant is b_0^3: dividing, then, both sides by b_0^3, we have

$$c_2 = \begin{vmatrix} \frac{a_2}{b_0}, & \frac{b_1}{b_0}, & \frac{b_2}{b_0} \\ \frac{a_1}{b_0}, & 1, & \frac{b_1}{b_0} \\ \frac{a_0}{b_0}, & 0, & 1 \end{vmatrix}$$

Similarly,

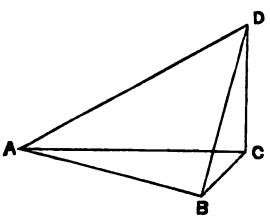
$$c_3 = \begin{vmatrix} \frac{a_3}{b_0}, & \frac{b_1}{b_0}, & \frac{b_2}{b_0}, & \frac{b_3}{b_0} \\ \frac{a_2}{b_0}, & 1, & \frac{b_1}{b_0}, & \frac{b_2}{b_0} \\ \frac{a_1}{b_0}, & 0, & 1, & \frac{b_1}{b_0} \\ \frac{a_0}{b_0}, & 0, & 0, & 1 \end{vmatrix};$$

and so on.

6. *Being given the lengths of three of the sides of a quadrilateral, find when its area is a maximum, and show that the construction of the quadrilateral is reducible to the solution of a cubic equation.*

Let ABCD be the maximum quadrilateral required: where AB, BC, CD are the given sides of lengths a, b, c respectively. It is clear that, by turning DC about C, we may alter the area of the quadrilateral without affecting the lengths of the given sides. But, since the area is a maximum, it must be incapable of increase by this process. Therefore DC must be at right angles to AC; for otherwise we could raise D by turning CD about C, and

Fig. 20.



thus increase the area of the triangle ACD, and therefore of the whole quadrilateral. Hence $\angle$ ACD is a right angle; and, similarly, $\angle$ ABD is a right angle, and a circle having AD for a diameter will pass round ABCD. To find AD, call its length x, and let the angles subtended by a, b, c at the centre of the circle be 2θ, 2ϕ, 2ψ; then $\theta+\phi+\psi=90°$, and hence

$$\sin^2\theta+\sin^2\phi+\sin^2\psi+2\sin\theta\sin\phi\sin\psi=1. \quad . \quad (1)$$

But $\qquad x\sin\theta=a, \quad x\sin\phi=b, \quad x\sin\psi=c;$

and therefore on substituting in (1), we have

$$x^3-x(a^2+b^2+c^2)-2abc=0. \quad . \quad . \quad . \quad (2)$$

This equation has two negative roots, as may be seen by writing for x successively 0, $-a$, $-\infty$; and therefore one positive root, which is that required. We can then describe the circle having x for diameter, and place in it the chords a, b, c in any order; for we notice that equation (2) is symmetrical in a, b, c.

7. *Prove that the reducing cubic of the equation*

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

may be written in the form of the determinant

$$\begin{vmatrix} a, & b, & y \\ b, & \frac{3c-y}{2}, & d \\ y, & d, & e \end{vmatrix}=0.$$

This result is obtained in Salmon's Higher Algebra, § 172, where $c-\lambda'$ is written instead of y.

8. *Prove De Moivre's Theorem, viz.*

$$(\cos\alpha+\sqrt{-1}.\sin\alpha)^n=\cos n\alpha+\sqrt{-1}.\sin n\alpha,$$

for all real values of n.

Find the simplest form of

$$\frac{(\cos x + \sqrt{-1}\,.\,\sin x)^4}{(\cos y - \sqrt{-1}\,.\,\sin y)^5}.$$

[Todhunter's Trigonometry, § 267.]

The given fraction is

$$\frac{\cos 4x + i \sin 4x}{\cos 5y - i \sin 5y} = \frac{(\cos 4x + i \sin 4x)(\cos 5y + i \sin 5y)}{1}$$
$$= \cos(4x+5y) + i \sin(4x+5y),$$

the simplest form required.

9. *Find the sum of the series*

$$\operatorname{cosec}\theta + \operatorname{cosec} 2\theta + \operatorname{cosec} 4\theta + \&c. \text{ to } n \text{ terms},$$

and

$$1 + \frac{x\cos\theta}{1} + \frac{x^2\cos 2\theta}{1.2} + \frac{x^3\cos 3\theta}{1.2.3} + \&c. \text{ ad infinitum.}$$

The first series is summed in Todhunter's Trigonometry, § 308; and a more general form of the second in the same book, § 311, the result being

$$\exp(x\cos\theta)\,.\,\cos(x\sin\theta).$$

10. *Find an expression for the radius of the circle inscribed in a spherical triangle in terms of its angles; and deduce from it an expression for the radius of the circumscribed circle in terms of the sides.*

[Todhunter's Spherical Trigonometry, §§ 89, 95.]

11. *In plane geometry the locus of a point such that the sum of the squares of its distances from any fixed points is constant, is a circle; and if the sum of its distances from any fixed right lines is constant, the locus is a right line. Show that both of these correspond to the same theorem in spherical geometry, and explain how the theorem on the sphere gives two different theorems on the plane.*

"Let A, B, C, ... be n fixed points on a sphere;

APX, BPY, CPZ, ... arcs meeting their polar circles in X, Y, Z, ... respectively; then the locus of the point P, when

$$\cos AP + \cos BP + \cos CP + .. = \text{constant}, \quad . \quad . \quad (1)$$

or

$$\sin PX + \sin PY + \sin PZ + .. = \text{constant}, \quad . \quad . \quad (2)$$

is a small circle which becomes a great circle when the constant is zero."

This theorem, which may be readily proved by solid geometry, and is implicitly proved in Todhunter's Spherical Trigonometry, § 170, gives both theorems in plane geometry by taking suitable limiting forms.

Taking equation (1), let r the radius of the sphere be large, and let the constant be $n - a \div r^2$. Expanding the cosines on the left, we have

$$1 - \frac{AP^2}{2r^2} + .. + 1 - \frac{BP^2}{2r^2} + .. + ... = n - \frac{a}{r^2},$$

or

$$\left.\begin{array}{r} \dfrac{AP^2 + BP^2 + CP^2 + ..}{2r^2} - \text{terms involving higher} \\ \text{powers of } r \text{ in the denominator} \end{array}\right\} = \frac{a}{r^2}.$$

Now make r infinite, AP, BP, .. remaining finite, we have then

$$AP^2 + BP^2 + CP^2 + .. = 2a;$$

and since P remains at a finite distance from A, B, C, .. we get the first theorem, the small circle on the sphere becoming a circle on the plane.

Again, taking equation (2), and assuming for the constant the value $a \div r$, we have

$$\frac{PX}{r} + \frac{PY}{r} + \frac{PZ}{r} + .. = \frac{a}{r},$$

or, in the limit,

$$PX + PY + PZ + .. = a.$$

Here PX, PY, PZ, .. must be supposed to remain finite (and A, B, C, .. go off to infinity); and the locus of P becomes ultimately a great circle, since the constant ulti-

mately vanishes; and since each infinite great circle becomes a straight line at a finite distance, we have the second theorem of the question.

NOTE.—The first theorem in plane geometry will reduce to the second if we take the fixed points at an infinite distance from the origin, and a finite distance from each other; for if (a_1, b_1), (a_2, b_2) . . be their coordinates, (x, y) those of P, the locus of P is

$$n(x^2+y^2)-2x\Sigma a-2y\Sigma b+\Sigma(a^2+b^2)=\text{const.},$$

and Σa, Σb are infinite; thus (x, y) must lie in a circle whose centre is at infinity, and which passes through the origin if the constant is $\Sigma(a^2+b^2)$; thus the position of the locus at a finite distance is a straight line.

12. *If* A′, B′, C′ *represent the angles formed by the chords of a spherical triangle* ABC, *prove the formula*

$$\frac{\sin(A'-B')}{\sin C'}=\frac{\sin(A-B)}{\sin C}.$$

From A draw the arc AD, making the angle BAD equal to B; then, if the plane through the centre O and AD meets the base of the chordal triangle in D′, it is clear that AD′=BD′, and thus the angle BAD′=B′. Hence

$$\angle CAD = A - B \text{ in the spherical triangle,}$$

$$\angle CAD' = A' - B' \text{ in the chordal triangle.}$$

Now

$$\frac{\sin(A'-B')}{\sin C'}=\frac{D'C}{D'A}=\frac{D'C}{D'B}=\frac{\sin D'OC}{\sin D'OB}$$

(from the isosceles triangle OBC)

$$=\frac{\sin DC}{\sin DB}=\frac{\sin DC}{\sin DA}=\frac{\sin(A-B)}{\sin C}.$$

Or we may proceed thus:—Let a', b', c' be the sides of the chordal triangle, r the radius of the sphere; then

$$a'r = 2r \sin \tfrac{1}{2}a, \quad b' = 2r \sin \tfrac{1}{2}b, \quad c' = 2r \sin \tfrac{1}{2}c.$$

Hence we have

$$\frac{\sin(A'-B')}{\sin C'} = \frac{\sin(A'-B')\sin(A'+B')}{\sin C' \sin(A'+B')}$$

$$= \frac{\sin^2 A' - \sin^2 B'}{\sin^2 C'} = \frac{a'^2 - b'^2}{c'^2} = \frac{\sin^2 \tfrac{1}{2}a - \sin^2 \tfrac{1}{2}b}{\sin^2 \tfrac{1}{2}c}$$

$$= \frac{\sin \tfrac{1}{2}(a+b)\sin \tfrac{1}{2}(a-b)}{\sin^2 \tfrac{1}{2}c}$$

$$= \frac{\sin \tfrac{1}{2}(A-B)\cos \tfrac{1}{2}(A-B)}{\sin \tfrac{1}{2}C \cos \tfrac{1}{2}C} \text{ (by Delambre's Analogies)}$$

$$= \frac{\sin(A-B)}{\sin C}.$$

Afternoon, 3 to 6.

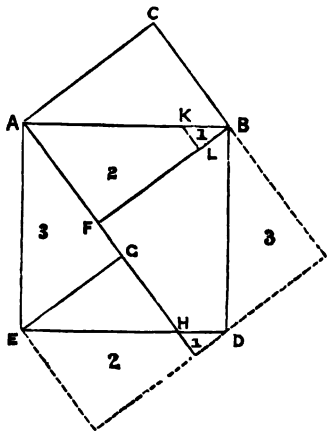
PURE AND COORDINATE GEOMETRY.

1. *Show how to cut the square on the hypothenuse of a right-angled triangle into five portions, so that they shall piece together to form two squares on the sides.*

Let AB be the hypothenuse of the right-angled triangle ABC; on AB describe the square ABDE; through A draw AH parallel to CB, the shortest side of the triangle; draw BF, EG parallel to AC, and make BK equal to HD, KL parallel to AH.

A little consideration will show that if the figures marked 1, 2, 3 be removed and applied to the remaining figure in the manner shown by the dotted lines, we shall obtain two squares whose sides are equal to those of the

Fig. 21.



triangle ABC; and the whole square is divided as required.

2. *Describe a circle which shall pass through a given point, and touch two given circles.*

This problem is solved in the Appendix to Todhunter's Euclid; and, in a more general form, in Salmon's Conic Sections, §§ 119, 121.

3. *Prove that the circle passing through the feet of the perpendiculars of a triangle also passes through the middle points of the sides.*

This well-known property of the nine points' circle is proved, amongst other places, in the Appendix to Todhunter's Euclid.

4. *The subnormal in a parabola is constant.*

PQ *is a chord of a parabola normal at* P; QR *is drawn parallel to the axis to meet the double coordinate* PP′ *pro-*

duced in R. *Prove that the rectangle contained by* PP′ *and* P′R *is constant.*

[Besant's Conic Sections, § 32.]

From Q draw the ordinate QM, and let PP′, PQ meet the axis in N, G respectively; then

$$PR \cdot P'R = RN^2 - PN^2 = QM^2 - PN^2 = 4AS(AM - AN) = 4AS \cdot RQ.$$

Hence

$$PR : RQ :: 4AS : P'R;$$

but $$PR : RQ :: PN : NG :: PP' : 2NG,$$

$$\therefore\ 4AS : P'R :: PP' : 2NG,$$

$$\therefore\ P'R \cdot PP' = 8AS \cdot NG = 16AS^2,$$

a constant quantity.

5. *Prove that the tangent at any point of a hyperbola bisects the angle between the focal distances of the point.*

If the normal meet the minor axis in g, *and if* S *be the focus, then* Pg *will be to* Sg *in a constant ratio.*

[Besant's Conic Sections, § 104.]

Let H be the other focus, and let Pg meet the major axis in G; then a circle will pass through S, H, P, g; and the triangles GPH, gPS are similar. Hence

$$Pg : Sg :: PH : GH :: CA : CS,$$

which is a constant ratio.

6. *Find the area enclosed by the three lines*

$$a_1x + b_1y + c_1 = 0,$$
$$a_2x + b_2y + c_2 = 0,$$
$$a_3x + b_3y + c_3 = 0,$$

the coordinate axes being oblique.

This is worked in Salmon's Conic Sections, § 39.

We may also proceed thus:—Let (x_1, y_1), (x_2, y_2), $(x_3 y_3)$ be the three angular points of the triangle, then its area is

$$A = \tfrac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \sin \omega,$$

where ω is the angle between the axes.

Now we have, from the given equations,

$$x_1(a_2b_3 - a_3b_2) = b_2c_3 - b_3c_2,$$

$$y_1(a_2b_3 - a_3b_2) = c_2a_3 - a_3c_2,$$

with similar equations for the other coordinates.

We may write these results

$$x_1C_1 = A_1, \quad y_1C_1 = B_1, \quad x_2C_2 = A_2, \text{ \&c.},$$

where A_1, A_2, A_3, &c. are the minors of the eliminant of the three given equations. Hence the area becomes

$$\frac{\tfrac{1}{2} \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \sin \omega}{C_1 C_2 C_3};$$

and the numerator being the reciprocal determinant (Salmon's Higher Algebra, § 30), we have finally

$$A = \frac{\tfrac{1}{2} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 \sin \omega}{(a_2b_3 - a_3b_2)(a_3b_1 - a_1b_3)(a_1b_2 - a_2b_1)}.$$

7. *If a quadrilateral be circumscribed about a conic, any diagonal is the polar of the intersection of the other two.*

[Besant's Conic Sections, § 199.]

8. *Find the angle between the tangents to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

from the point (X, Y)*; and deduce the locus of the point* (X, Y) *when this angle is constant.*

Let $y=mx+\sqrt{m^2a^2+b^2}$ be taken as the equation to a tangent through (X, Y); then the two values of m are given by

$$Y=mX+\sqrt{a^2m^2+b^2},$$

or
$$(X^2-a^2)m^2-2mXY+(Y^2-b^2)=0. \quad . \quad . \quad . \quad (1)$$

Now, if m_1, m_2 are the roots of (1), the angle θ between the two tangents from (X, Y) is given by the equation

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}.$$

Now

$$m_1+m_2=\frac{2XY}{X^2-a^2}, \quad m_1m_2=\frac{Y^2-b^2}{X^2-a^2}, \text{ from (1).}$$

Thus

$$\tan^2\theta=\frac{(m_1+m_2)^2-4m_1m_2}{(1+m_1m_2)^2}$$

$$=\frac{4(a^2Y^2+b^2X^2-a^2b^2)}{(X^2+Y^2-a^2-b^2)^2}.$$

If θ is constant, the locus of (X, Y) is given by the equation

$$(X^2+Y^2-a^2-b^2)^2\tan^2\theta=4(a^2Y^2+b^2X^2-a^2b^2),$$

a curve of the fourth degree, except when $\theta=\frac{1}{2}\pi$, in which case it becomes the director circle of the ellipse.

9. *Find the locus of the points of contact of tangents to a series of confocal ellipses from a fixed external point, noticing the case when the point is on one of the coordinate axes.*

Let $2c$ represent the distance between the foci; then the equation to any ellipse of the series is

$$\frac{x^2}{c^2+\lambda^2}+\frac{y^2}{\lambda^2}=1, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where λ is its (arbitrary) semi-axis minor.

Now, if tangents be drawn from (h, k) to (1), their points of contact will lie on

$$\frac{xh}{c^2+\lambda^2}+\frac{yk}{\lambda^2}=1. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Eliminating λ^2 between (1) and (2), we readily obtain for the locus of the points of contact

$$c^2(x-h)(y-k)=(yh-xk)\{x(x-h)+y(y-k)\},$$

a curve of the third degree, passing through (h, k).

In the special case, when one coordinate of the given point vanishes, *e. g.* when $k=0$, the equation becomes

$$c^2(x-h)=h\{x(x-h)+y^2\},$$

which represents a circle.

10. *Prove Pascal's Theorem, viz. that the three intersections of the opposite sides of any hexagon inscribed in a conic are in one right line.*

[Salmon's Conic Sections, § 267.]

11. *Two sides of a triangle inscribed in a conic pass respectively through two fixed points. Find the envelope of its base.*

This problem is solved by three distinct methods in Salmon's Conic Sections, §§ 272, 304, 354.

Tuesday, August 3, 1880.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

Morning, 10 *to* 1.

DIFFERENTIAL CALCULUS.

1. *Starting from first principles, deduce the differential coefficients of* x^n *for any index, and of* $\log_a x$.

[Williamson's Differential Calculus, §§ 16, 17, 18, 29.]

2. *Prove Taylor's expansion for* $f(x+h)$, *and deduce Lagrange's expression for the remainder after any number of terms of the series.*

[Williamson's Differential Calculus, §§ 55, 75.]

3. *Express* $\sec(x+iy)$ *and* $\sec^{-1}(x-iy)$ *in the form* $u+iv$, *where* u *and* v *are rational functions of* x *and* y; i *representing* $\sqrt{-1}$.

We have seen [page 108, qu. 4] that $\cos(x+iy)$ is equal to $\cos x \cosh y - i \sin x \sinh y$; therefore we have

$$\sec(x+iy) = \frac{1}{\cos x \cosh y - i \sin x \sinh y}$$
$$= \frac{\cos x \cosh y + i \sin x \sinh y}{\cos^2 x \cosh^2 y + \sin^2 x \sinh^2 y},$$

which is of the form required.

Again, if $\sec^{-1}(x-iy) = u - iv$,

we have $\sec(u-iv) = x - iy$,

or $\cos(u-iv) = \frac{x+iy}{x^2+y^2}$.

Calling $x^2+y^2 r^2$, and equating the real and imaginary parts of this identity, we get

$$r^2 \cos u \cos iv = x,$$
$$r^2 \sin u \sin iv = iy.$$

From these equations, which are symmetrical in u and iv, we find that u and iv are roots of the equation

$$x^2 \sin^2\theta - y^2 \cos^2\theta = r^4 \sin^2\theta \cos^2\theta;$$

and solving, we find

$$2r^2 \cos^2\theta = r^2+1 \pm \sqrt{(r^2-1)^2+4y^2},$$
$$= r^2+1 \pm \sqrt{(r^2+1)-4x^2}.$$

Thus there are two values of $\cos^2\theta$, one greater, the other less than unity: the first of these gives $\cos^2 iv$, or $\cosh^2 v$, the second gives $\cos^2 u$. In fact

$$2r \cos u = \sqrt{(r^2+2x+1)} - \sqrt{(r^2-2x+1)},$$
$$2r \cosh v = \sqrt{(r^2+2x+1)} + \sqrt{(r^2-2x+1)},$$

and u is found as an inverse circular function of x and y, while v is a logarithmic function; for $\cosh v$ is

$$\tfrac{1}{2}\{\exp v + \exp(-v)\}.$$

4. *Prove that* $n \tan \frac{\pi}{n}$ *diminishes as* n *increases, provided* n *be greater than* 2. *Hence, or otherwise, prove that if two regular polygons have equal perimeters, that which has the greater number of sides has the greater area.*

Let $y = n \tan \frac{\pi}{n}$, then

$$\frac{dy}{dn} = \tan\frac{\pi}{n} - \frac{\pi}{n}\sec^2\frac{\pi}{n} = \tfrac{1}{2}\sec^2\frac{\pi}{n}\left(\sin\frac{2\pi}{n} - \frac{2\pi}{n}\right),$$

which is negative when n is greater than 2: thus the given expression decreases as n increases.

Again, the area of a polygon of n sides, and of perimeter l, is easily found to be $\frac{l^2}{2n}\cot\frac{\pi}{n}$, which, by the result already obtained, increases with n.

5. *If the number of sides of a regular polygon inscribed in a circle be so great that the length of each side is very small in comparison with the radius of the circle, prove that the difference between the circumference of the circle and the perimeter of the polygon is very small in comparison with the length of a side of the polygon.*

Let 2α be the small angle which each side subtends at the centre of the circle, a the radius of the circle; then $n\alpha=\pi$, and the difference between the perimeters of the circle and polygon is $2a(\pi-n\sin\alpha)$ or $2an(\alpha-\sin\alpha)$. The ratio of this to a side of the polygon is $n(\alpha-\sin\alpha):\sin\alpha$, which, as we have to prove, is very small.

Now $\alpha-\sin\alpha$ is of the third order in the small quantity α; and since $n\alpha$ is finite, it follows that $n(\alpha-\sin\alpha)$ is of the second order, and $\sin\alpha$ is of the first order; thus

$$n(\alpha-\sin\alpha):\sin\alpha$$

is of the first order, and is therefore very small.

6. *Find the limiting value of*

$$\frac{x^n\sin ny-y^n\sin nx}{\tan nx-\tan ny}$$

when $y=x$; *and of*

$$\frac{2\cot x}{1-\cos x}\tan^{-1}(m\tan x)-m\cot^2\frac{x}{2}\ \textit{when}\ x=0.$$

In the first expression suppose y to approach the value x; it takes the form $0\div0$; and differentiating with regard to y in the usual manner, the limit is that of

$$\frac{nx^n\cos ny-ny^{n-1}\sin nx}{-n\sec^2 ny}\ \text{when}\ y=x,\ \text{or}$$

$$x^{n-1}(\sin nx-x\cos nx)\cos^2 nx.$$

The second expression is best evaluated by expansion; it becomes

$$\{\cot x(m\tan x - \tfrac{1}{3}m^3\tan^3 x + \ldots) - m\cos^2\tfrac{1}{2}x\} \div \sin^2\tfrac{1}{2}x,$$

or $$\{m - \tfrac{1}{3}m^3\tan^2 x + \ldots - m + m\sin^2\tfrac{1}{2}x\} \div \sin^2\tfrac{1}{2}x,$$

or, in the limit, $m - \frac{4}{3}m^3$.

7. *Of all pyramids standing on a triangular base and having a given volume, find that whose surface is a minimum, and determine the minimum area.*

Let ABC be the triangular base; then all pyramids on it, and of constant volume, must be of the same height h. Let P be any vertex, PH perpendicular to the plane ABC, PL perpendicular to BC; then LH is clearly perpendicular to BC, and is one of the trilinear coordinates, α, of the point H, referred to the triangle ABC.

Now the area of the triangle PBC is $\frac{1}{2}$PL . BC, and PL$=\sqrt{(\alpha^2+h^2)}$, BC$=a$; hence the area of PBC is $\frac{1}{2}a\sqrt{(\alpha^2+h^2)}$, with similar expressions for the areas of PCA, PAB. Hence the area of the pyramid is

$$\tfrac{1}{2}a\sqrt{(\alpha^2+h^2)} + \tfrac{1}{2}b\sqrt{(\beta^2+h^2)} + \tfrac{1}{2}c\sqrt{(\gamma^2+h^2)} + S; \quad (1)$$

where S is the area of the triangle ABC; and

$$a\alpha + b\beta + c\gamma = 2S. \quad \ldots\ldots\ldots \quad (2)$$

To make (1) a minimum we have

$$\frac{a\alpha\,d\alpha}{\sqrt{(\alpha^2+h^2)}} + \frac{b\beta\,d\beta}{\sqrt{(\beta^2+h^2)}} + \frac{c\gamma\,d\gamma}{\sqrt{(\gamma^2+h^2)}} = 0,$$

$$a\,d\alpha + b\,d\beta + c\,d\gamma = 0,$$

for all small variations in α, β, γ.

This implies the equality

$$\frac{\alpha^2+h^2}{\alpha^2} = \frac{\beta^2+h^2}{\beta^2} = \frac{\gamma^2+h^2}{\gamma^2},$$

and therefore $\alpha=\beta=\gamma$; or P must be vertically above the centre of the inscribed circle of the base ABC; and the area required is found to be $\sqrt{(S^2+h^2s^2)}+S$.

8. *Explain the method of finding the asymptotes of an algebraic curve, and determine the asymptotes to the curve*

$$x^3+x^2y-6xy^2-x^2+13xy-12y^2-6x+3y-10=0.$$

[Williamson's Differential Calculus, ch. xiii.]

We have in the example given,

$$x^3+x^2y-6xy^2=x(x-2y)(x+3y). \quad . \quad . \quad (1)$$

The asymptotes are therefore of the forms

$$x+a, \quad x-2y+b, \quad x+3y+c;$$

and the equation of the curve is therefore

$$(x+a)(x-2y+b)(x+3y+c)+lx+my+n=0.$$

Equating terms of the second degree, we have

$$-x^2+13xy-12y^2=a(x-2y)(x-3y)+bx(x-3y)+cx(x-2y),$$

$$\therefore \quad \frac{-x^2+13xy-12y^2}{x(x-2y)(x+3y)}=\frac{a}{x}+\frac{b}{x-2y}+\frac{c}{x+3y}.$$

On resolving the left-hand member into partial fractions, we find

$$a=2, \quad b=1, \quad c=-4;$$

and the asymptotes are therefore

$$x+2=0, \quad x-2y+1=0, \quad x+3y-4=0;$$

while the line on which their other intersections with the curve lie, viz. $lx+my+n=0$, is found to be

$$4x-19y-2=0.$$

9. *Eliminate, by differentiation, the arbitrary function from the equation*

$$z=x^my^n+\phi(x+y)+\psi(x-y).$$

Here
$$p=mx^{m-1}y^n+\phi'+\psi',$$
$$q=nx^my^{n-1}+\phi'-\psi',$$

using the usual notation.

Hence $r=m(m-1)x^{m-2}y^n+\phi''+\psi''$,

$t=n(n-1)x^my^{n-2}+\phi''+\psi''$;

$\therefore \quad r-t=x^{m-2}y^{n-2}\{m(m-1)y^2-n(n-1)x^2\}$,

the result of the elimination required.

10. *Investigate the points of inflexion on the curve*

$$r=a+b\cos n\theta;$$

and show that it has no real points of inflexion unless

$$a \text{ be } >b \text{ and } <(1+n^2)b.$$

At a point of inflexion the radius of curvature is infinite. Now the denominator of the fraction giving the radius of curvature in polar coordinates is

$$r^2-r\frac{d^2r}{d\theta^2}+2\left(\frac{dr}{d\theta}\right)^2,$$

which must therefore vanish at a point of inflexion.

Expressing all in terms of θ, we have

$$(a+b\cos n\theta)^2+bn^2\cos n\theta(a+b\cos n\theta)+2b^2n^2\sin^2 n\theta=0.$$

Now writing $\cos n\theta=t$, and reducing, we have to find when the equation

$$f(t)=t^2b^2(1-n^2)+abt(2+n^2)+a^2+2b^2n^2=0$$

has a root between $+1$ and -1. Now, by applying the usual test, we find that this equation has no real roots unless $a^2(8+n^2)>8b^2(1-n^2)$. In this case it may be easily shown that $a^2+2b^2n^2>b^2(1-n^2)$; and the product of the roots is therefore greater than unity, and there can be, at most, but one root between $+1$ and -1. If there is a root, $f(1)$ and $f(-1)$ must have contrary signs. Now

$$f(1)=(a+b)(a+b+bn^2),$$

which is positive; and

$$f(-1)=(a-b)(a-b-bn^2),$$

which must therefore be negative. The first factor is the greater and must therefore be the positive one, and thus

$$a>b, \text{ and } <(1+n^2)b.$$

11. *Prove that the evolute of an epicycloid is a similar epicycloid.*

[Williamson's Differential Calculus, § 281.]

12. *Find the radii of curvature of the two branches of the curve, $y^3 = ax(bx+cy)$, at the origin, and trace the curve.*

Solving for x, and expanding the surd, we have

$$x = -\frac{cy}{2b} \pm \left(\frac{cy}{2b} + \frac{y^2}{ac} + \ldots\right);$$

thus, at the origin,

$$\frac{dx}{dy} = 0 \text{ or } -\frac{c}{b}, \quad \frac{d^2x}{dy^2} = \pm\frac{2}{ac}.$$

Hence the radii of curvature are respectively

$$\frac{ac}{2}, \text{ and } \frac{ac}{2}\left(1+\frac{c^2}{b^2}\right)^{\frac{3}{2}}.$$

Putting $a=1$ to make the equation homogeneous, the exact value of x is easily seen to be given by

$$2bx = -cy \pm y\sqrt{c^2+4by}.$$

Thus y has a minimum value, viz. $-\frac{c^2}{4b}$; and x has always two values for a given value of y, giving two points equidistant from the line $2bx+cy=0$.

The tangents at the origin, which is a double point, are $x=0$ and $bx+cy=0$. There are clearly no asymptotes; and the curve consists of a loop lying between the negative direction of the axis of y and $bx+cy=0$, whose boundaries proceed respectively, after passing the node, into the quadrants where y is positive.

Afternoon, 3 *to* 6.

INTEGRAL CALCULUS.

1. *Find the integrals of*

$$\log(x+\sqrt{1+x^2})dx \quad \textit{and} \quad \frac{dx}{x+\sqrt{1+x^2}}.$$

Integrating by parts, we have

$$\int \log(x+\sqrt{1+x^2})dx = x\log(x+\sqrt{1+x^2}) - \int \frac{xdx}{\sqrt{1+x^2}}$$
$$= x\log(x+\sqrt{1+x^2}) - \sqrt{1+x^2}.$$

Again,

$$\int \frac{dx}{x+\sqrt{1+x^2}} = \int(\sqrt{1+x^2}-x)dx$$
$$= \tfrac{1}{2}x\sqrt{1+x^2} + \tfrac{1}{2}\log(x+\sqrt{1+x^2}) - \tfrac{1}{2}x^2.$$

2. *Integrate*

$$\frac{(1+x\cos\alpha)dx}{(1+2x\cos\alpha+x^2)^{\frac{3}{2}}} \quad \textit{and} \quad \sin^{\frac{1}{2}}\theta\cos^5\theta d\theta.$$

On writing $x=\frac{1}{y}$, the first integral becomes

$$-\int \frac{(y+\cos\alpha)dy}{(y^2+2y\cos\alpha+1)^{\frac{3}{2}}} = (y^2+2y\cos\alpha+1)^{-\frac{1}{2}}$$
$$= \frac{x}{\sqrt{1+2x\cos\alpha+x^2}}.$$

Again,

$$\int \sin^{\frac{1}{2}}\theta\cos^5\theta d\theta = \int \sin^{\frac{1}{2}}\theta(1-\sin^2\theta)^2\cos\theta d\theta,$$

which, on expanding and integrating, becomes

$$\frac{2}{7}\sin^{\frac{7}{2}}\theta - \frac{4}{11}\sin^{\frac{11}{2}}\theta + \frac{2}{15}\sin^{\frac{15}{2}}\theta.$$

3. *Find the value of the definite integrals*

$$\int_0^\pi \frac{dx}{(1-2a\cos x+a^2)(1-2b\cos x+b^2)} \text{ and } \int_0^{\frac{\pi}{2}} \sin^m x \cos^n x dx,$$

where m and n are integers.

In the first example we may readily split up the subject of integration into partial fractions; we get

$$\int_0^\pi \left\{ \frac{adx}{1-2a\cos x+a^2} - \frac{bdx}{1-2b\cos x+b^2} \right\} \div (a-b)(1-ab).$$

Each integral may be evaluated by means of the result in Williamson's Integral Calculus, § 18; and we obtain

$$\left(\frac{\pi a}{1-a^2} - \frac{\pi b}{1-b^2} \right) \div (a-b)(1-ab),$$

or

$$\frac{\pi(1 \dot{+} ab)}{(1-a^2)(1-b^2)(1-ab)}.$$

For the second example see Williamson, pp. 120, 121.

4. *Show that the integral*

$$\int F(x, \sqrt{a+2bx+cx^2})dx,$$

in which F *is a rational algebraic function, is always reducible to the integral of a rational algebraic expression.* [Williamson, § 78.]

5. *Find the value of*

$$\int_0^\infty \frac{[(\tan^{-1}ax)^2-(\tan^{-1}bx)^2]dx}{x}.$$

Consider the integral

$$\int_0^\infty [\phi(x)-\phi(mx)]\frac{dx}{x} = u.$$

On writing for x in succession mx, m^2x, ... &c., we have

$$u=\int_0^\infty [\phi(x)-\phi(mx)]\frac{dx}{x},$$

$$u=\int_0^\infty [\phi(mx)-\phi(m^2x)]\frac{dx}{x},$$

$$\ldots = \ldots\ldots$$

$$u=\int_0^\infty [\phi(m^{n-1}x)-\phi(m^nx)]\frac{dx}{x}.$$

Adding,

$$nu=\int_0^\infty [\phi(x)-\phi(m^nx)]\frac{dx}{x}$$

$$=n\int_0^\infty [\phi(x)-\phi(mx)]\frac{dx}{x}.$$

Now, let $m^n=c$; and let n become infinite, while c remains constant and m approaches unity. We have

$$\int_0^\infty [\phi(x)-\phi(cx)]\frac{dx}{x}=\int_0^\infty \frac{[\phi(x)-\phi(c^{\frac{1}{n}}x)]}{\frac{1}{n}}\frac{dx}{x};$$

and in the limit, since the right-hand expression is of the form $0\div 0$, it becomes, on differentiating with regard to $\frac{1}{n}$,

$$-\int_0^\infty \phi'(x)\log c\,dx=\log c[\phi(0)-\phi(\infty)].$$

In the given example,

$$ca=b,\quad \phi(x)=(\tan^{-1}ax)^2,\quad \phi(0)=0,\quad \phi(\infty)=\tfrac{1}{4}\pi^2;$$

and the result is therefore

$$\tfrac{1}{4}\pi^2(\log a-\log b).$$

6. *Find the area of the segment cut off from a parabola by a focal chord in terms of the length of the chord and the parameter of the parabola.*

Let PSQ be a focal chord, T its pole, X the foot of the directrix, TV parallel to the axis. Then, since TS is perpendicular to PQ, from similar triangles $TV \cdot SX = TS^2$.

Now, let $SX = 2a$, $PQ = c$; then $TV = \frac{1}{2}c$, and the area of the segment is

$$\tfrac{2}{3} \cdot \Delta TPQ = \tfrac{1}{3}TS \cdot PQ = \tfrac{1}{3}c\sqrt{TV \cdot SX} = \tfrac{1}{3}c\sqrt{ca},$$

the result required.

7. *In a curve, if the perpendicular from the origin on the tangent varies directly as the* nth *power of the radius vector drawn to the point of contact, find the equation of the curve in polar coordinates.*

We may write our given equation in the form

$$pc^{n-1} = r^n,$$

except when $n = 1$.

Now
$$p^2\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\} = r^4;$$

hence
$$\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\} r^{2n-4} = c^{2n-2},$$

and
$$\frac{d\theta}{dr} = \frac{r^{n-2}}{\sqrt{(c^{2(n-1)} - r^{2(n-1)})}};$$

whence
$$(n-1)\theta = \sin^{-1}\left(\frac{r^{n-1}}{c^{n-1}}\right),$$

no constant being needed, since the initial line is abitrary. The equation to the curve thus becomes

$$r^{n-1} = c^{n-1}\sin(n-1)\theta.$$

When $n = 1$ this form fails; but taking the equation $p = \mu r$ we readily obtain an equiangular spiral.

8. *Prove that the difference between the length of an asymptote and the corresponding infinite arc of a hyperbola is equal to the difference between the lengths of the quadrants of two ellipses.*

[Williamson's Integral Calculus, § 160.]

9. *Find the value of*

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x \, dx\, dy}{\sqrt{1-\sin^2 x \sin^2 y}}.$$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\sin x \cos x \, dx\, dy}{\sqrt{1-\sin^2 x \sin^2 y}} = -\int_0^{\frac{\pi}{2}} \left[\frac{\sqrt{1-\sin^2 x \sin^2 y}}{\sin^2 y}\right]_0^{\frac{\pi}{2}} dy$$

$$= \int_0^{\frac{\pi}{2}} \frac{(1-\cos y)dy}{\sin^2 y} = \int_0^{\frac{\pi}{2}} \tfrac{1}{2}\sec^2 \tfrac{1}{2}y \, dy$$

$$= \left[\tan \tfrac{1}{2}y\right]_0^{\frac{\pi}{2}} = 1.$$

10. *Find the volume of the surface which is represented by the equation*

$$\left(\frac{x}{a}\right)^{\frac{4}{3}} + \left(\frac{y}{b}\right)^{\frac{4}{3}} + \left(\frac{z}{c}\right)^{\frac{4}{3}} = 1.$$

The volume required is given by $\iiint dx\, dy\, dz$ taken throughout the given surface. Writing $x = a\xi^3$, $y = b\eta^3$, $z = c\zeta^3$, we now require

$$\iiint 125\, abc\, \xi^4 \eta^4 \zeta^4 \, d\xi\, d\eta\, d\zeta$$

taken throughout the surface $\xi^2 + \eta^2 + \zeta^2 = 1$.

Transforming to polar coordinates, we have

$$125\, abc \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_0^{2\pi}\int_0^1 r^2 \cos^9\theta \sin^4\theta \cos^4\phi \sin^4\phi \, dr\, d\phi\, d\theta,$$

which easily reduces to $\dfrac{100\,\pi abc}{3 \cdot 7 \cdot 11 \cdot 13}$.

11. *One edge of a right cylinder passes through the centre of a sphere, the radius of the cylinder being less than half that of the sphere. Find an expression for the portion of the surface of the cylinder between the sphere.*

Let a be the radius of the sphere, b the diameter of the cylinder. Through O, the centre of the sphere, draw a plane perpendicular to the axis of the cylinder; this plane is also perpendicular to all the generating lines, and it therefore bisects them. Let it meet one of them in P, and let OP make an angle θ with the tangent to the circular section of the cylinder at O: then $OP = b \sin\theta$, and the length of the generating line is $2\sqrt{(a^2 - b^2\sin^2\theta)}$. Now the elementary arc of the circular section is $2bd\theta$, and the elementary area of the cylinder between two generating lines is therefore $4b\sqrt{a^2 - b^2\sin^2\theta}\, d\theta$, and the whole area is

$$4\int_0^{\pi} b\sqrt{a^2 - b^2\sin^2\theta}\, d\theta,$$

an elliptic integral of the second class.

12. *Find the value of the mean square of the distance between two points taken at random within the circumference of a given circle.*

Let the equation of the circle be $x^2+y^2=a^2$, and let any pair of points within it be (x, y), (ξ, η); then we require the value of the expression

$$\frac{\iiiint \{(x-\xi)^2+(y-\eta)^2\}\, dx\, dy\, d\xi\, d\eta}{\iiiint dx\, dy\, d\xi\, d\eta},$$

the integrals being taken all over the area of the circle.

The numerator is

$$\iiiint (x^2+y^2+\xi^2+\eta^2)\, dx\, dy\, d\xi\, d\eta,$$

since the integrals of the other terms vanish from symmetry.

This expression becomes on integration,

$$\pi a^2 \iint (x^2+y^2)dx\,dy + \pi a^2 \iint (\xi^2+\eta^2)d\xi\,d\eta,$$

or
$$2\pi a^2 \iint (x^2+y^2)\,dx\,dy,$$

which is $2\pi a^2 \iint r^3\,dr\,d\theta$ or $\pi^2 a^4$.

The denominator is $\pi^2 a^2$; thus the mean value required is

$$\pi^2 a^4 \div \pi^2 a^2 \quad \text{or} \quad a^2.$$

We may also solve the problem thus :—Let P, Q be points at distances r, r' from the centre O, and let $\angle POQ = \theta$; then

$$PQ^2 = r^2 + r'^2 - 2r\,r' \cos\theta.$$

Now, either by evaluating

$$\int_0^{2\pi} (r^2 + r'^2 - 2r\,r' \cos\theta)d\theta \div \int_0^{2\pi} d\theta,$$

or by producing OQ backwards to Q' so that $OQ' = r'$, and then considering together the pairs of opposite positions of Q, Q', we find that the mean value of PQ^2 for assigned values of OP, OQ is $r^2 + r'^2$.

Now the number of cases in which OP has the value r is proportional to the circumference of a circle of radius r, or to r; and similarly for OQ. Thus our final mean value will be represented by the expression

$$\frac{\iint r\,r'(r^2+r'^2)dr\,dr'}{\iint r\,r'\,dr\,dr'},$$

of which the denominator is proportional to the whole number of cases. The limits of each integration being 0, a, the result obtained is, as before, a^2.

Monday, August 1, 1881.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
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Morning, 10 *to* 1.

ALGEBRA AND TRIGONOMETRY.

1. *Solve the equations*

$$\sin\theta-\cos\theta-4\sin\theta\cos^2\theta=0;$$

$$\sqrt{\frac{1}{a^2}-\frac{1}{x^2}}+\sqrt{\frac{1}{b^2}-\frac{1}{x^2}}+\sqrt{\frac{1}{c^2}-\frac{1}{x^2}}=\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{x^2}}.$$

In the first equation we at once obtain

$$\sin\theta-\cos\theta-4\sin\theta(1-\sin^2\theta)=0,$$

or $$3\sin\theta-4\sin^3\theta+\cos\theta=0,$$

or $$\sin 3\theta+\cos\theta=0,$$

or $$\sin 3\theta=\sin\left(\theta-\frac{\pi}{2}\right);$$

whence $$3\theta=n\pi+(-1)^n\left\{\theta-\frac{\pi}{2}\right\},$$

the least positive root being $\theta=\frac{3\pi}{8}$.

In the second equation, writing p for $\frac{1}{a^2}$, q for $\frac{1}{b^2}$, r for $\frac{1}{c^2}$, and y for $\frac{1}{x^2}$, we have

$$\sqrt{(p-y)}+\sqrt{(q-y)}+\sqrt{(r-y)}=\sqrt{(p+q+r-y)}.$$

Transposing and squaring,

$$p+q-2y+2\sqrt{(p-y)(q-y)}=p+q+2r-2y-2\sqrt{(r-y)(p+q+r-y)},$$

or $$\sqrt{(p-y)(q-y)}-r=-\sqrt{(r-y)(p+q+r-y)}.$$

Squaring and reducing,

$$-2r\sqrt{(p-y)(q-y)}=rp+rq-pq-2ry.$$

Again squaring and reducing,

$$4pqry=2pqr(p+q+r)-q^2r^2-r^2p^2-p^2q^2.$$

Restoring the values of p, q, r, y, in terms of a^2, b^2, c^2, x^2, we find that

$$x=\pm\frac{2abc}{\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}}$$

$$=\pm\frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}},$$

where $$2s=a+b+c.$$

[See remark on qu. 1, page 124.]

The form of this result, together with that of the original equation, suggests the substitution

$$\pm x=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

in the given equation. It thus becomes

$$\pm\frac{\cos A}{a}\pm\frac{\cos B}{b}\pm\frac{\cos C}{c}=\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\frac{1}{x^2}}; \quad . \quad . \quad . \quad (2)$$

and, by taking one of two supplementary values of A, B, C which satisfy (1), we may ensure that all the signs on the left of (2) are positive; it becomes by virtue of relations (1),

$$\cot A+\cot B+\cot C=\sqrt{\operatorname{cosec}^2A+\operatorname{cosec}^2B+\operatorname{cosec}^2C-1};$$

or, on squaring and reducing,

$$\tan A+\tan B+\tan C=\tan A\tan B\tan C,$$

or $$A+B+C=\pi, \quad \text{or} \quad 2\pi;$$

since A, B, C may all be assumed positive and less than π. Thus A, B, C, or their supplements, are the angles of a triangle whose sides are a, b, c; and hence, by (1), $\pm x=$ diameter of circumscribing circle, which agrees with the

form already obtained. The actual signs to be taken on the left of the original equation are given by the values of cos A, cos B, cos C.

2. *If the* nth *term of a series be* n^3-2n^2+3n-2, *find its sum to n terms.*

$$\begin{aligned} n^3-2n^2+3n-2 &= (n-1)(n^2-n+2) \\ &= (n-1)(n(n+1)-2n+2) \\ &= (n-1)n(n+1)-2(n-1)n+2(n-1). \end{aligned}$$

The terms are now arranged as factorials; and the sum to n terms is therefore

$$\frac{(n-1)n(n+1)(n+2)}{4}-\frac{2(n-1)n(n+1)}{3}+\frac{2(n-1)n}{2},$$

no constant being needed.

The result reduces to

$$\frac{(n-1)n(3n^2+n+10)}{12}.$$

See Todhunter's Algebra, § 661.

3. *Find the limiting values of*

$$x\sqrt{x^2+a^2}-\sqrt{x^4+a^4} \quad \textit{when } x=\infty,$$

and of

$$\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}} \quad \textit{when } x=a.$$

The first expression is

$$x\sqrt{x^2+a^2}-\sqrt{x^4+a^4}=\frac{a^2(x^2-a^2)}{x\sqrt{x^2+a^2}+\sqrt{x^4+a^4}};$$

and when $x=\infty$ this becomes

$$\frac{a^2x^2-\text{lower terms}}{x^2+x^2+\text{lower terms}}=\frac{a^2}{2}.$$

The same result may be obtained by the binomial theorem, as follows:—

$$x\sqrt{x^2+a^2}-\sqrt{x^4+a^4}=x^2\left(\sqrt{1+\frac{a^2}{x^2}}-\sqrt{1+\frac{a^4}{x^4}}\right);$$

the expansions are both convergent, since in our limiting case $\frac{a}{x}$ is small. We have then

$$x^2\left(1+\frac{a^2}{2x^2}-\frac{a^4}{8x^4}-\ldots.\right)-x^2\left(1+\frac{a^4}{2x^4}-\frac{a^8}{8x^8}-\ldots.\right)$$

$$=\frac{a^2}{2}-\frac{5a^4}{8x^2}-\ldots.=\frac{a^2}{2} \text{ when } x=\infty.$$

Again,

$$\frac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}=\frac{(a+2x-3x)(\sqrt{3a+x}+2\sqrt{x})}{(3a+x-4x)(\sqrt{a+2x}+\sqrt{3x})}$$

$$=\tfrac{1}{3}\cdot\frac{\sqrt{3a+x}+2\sqrt{x}}{\sqrt{a+2x}+\sqrt{3x}} \text{ always.}$$

Now put $x=a$, and the value of the given expression in the limit is

$$\tfrac{1}{3}\cdot\frac{\sqrt{4a}+2\sqrt{a}}{\sqrt{3a}+\sqrt{3a}}=\frac{4}{6\sqrt{3}}=\frac{2\sqrt{3}}{9}.$$

4. *If* α, β, γ, *&c. be the roots of the equation*

$$ax^n+bx^{n-1}+cx^{n-2}+dx^{n-3}+\&c.=0,$$

find, in terms of the coefficients, $\Sigma\alpha^2\beta\gamma$.

We have, from the Theory of Equations,

$$\Sigma\alpha\beta\gamma=-\frac{d}{a},$$

$$\Sigma\alpha\ \ =-\frac{b}{a}.$$

Multiplying these together, the left-hand clearly becomes $\Sigma a^2\beta\gamma + 4\Sigma a\beta\gamma\delta$. Hence

$$\Sigma a^2\beta\gamma + 4\Sigma a\beta\gamma\delta = \frac{bd}{a^2};$$

but

$$\Sigma a\beta\gamma\delta = \frac{e}{a};$$

hence

$$\Sigma a^2\beta\gamma = \frac{bd}{a^2} - \frac{4e}{a} = \frac{bd - 4ea}{a^2}.$$

5. *Solve the equation*

$$x^{2n} + 2px^n + q = 0,$$

giving expressions for all its roots.

We have

$$x^n = -p \pm \sqrt{p^2 - q}.$$

CASE 1.—Let $p^2 - q$ be positive, say a^2; then $x^n = -p \pm a$; and if p, q are positive (which case alone we shall consider), both the values of x^n are negative, say $-r^n$, $-s^n$. Then we have

$$x^n = -r^n = r^n(\cos\pi + i\sin\pi),$$

where $i = \sqrt{-1}$.

Thus

$$x = r(\cos\pi + i\sin\pi)^{\frac{1}{n}},$$

$$= r\left(\cos\frac{2k+1}{n}\pi + i\sin\frac{2k+1}{n}\pi\right),$$

where k is any positive integer from 1 to n; similarly for the value $x^n = -s^n$. And all the roots in this case are given by

$$x = \left(p \pm \sqrt{p^2 - q}\right)^{\frac{1}{n}}\left(\cos\frac{2k+1}{n}\pi + i\sin\frac{2k+1}{n}\pi\right).$$

CASE 2.—Let p^2-q be negative and equal to $-b^2$.

We have $$x=-p\pm ib;$$

and, by proceeding as before, we have

$$x=(p^2+b^2)^{\frac{1}{2n}}\left(\cos\frac{2k\pi+\theta}{n}\pm i\sin\frac{2k\pi+\theta}{n}\right),$$

where $\tan\theta=-\frac{b}{p}$.

6. *Determine the condition that the equation*

$$ax^4+4bx^3+6cx^2+4dx+e=0$$

should have two equal roots; and prove that, if it has three equal roots,

$$ae-4bd+3c^2=0,$$
$$ace+2bcd-ad^2-eb^2-c^3=0.$$

The condition for two equal roots is given in Salmon's Higher Algebra, § 207, and is

$$D=(ae-4bd+3c^2)^3-27(ace+2bcd-ad^2-eb^2-c^3)^2=0.$$

If there are three equal roots, the first two derived functions of the given equation must have a root in common with it; in other words,

$$ax^4+4bx^3+6cx^2+4dx+e=0, \quad . \ . \ . \quad (1)$$
$$ax^3+3bx^2+3cx+d=0, \quad . \ . \ . \quad (2)$$
$$ax^2+2bx+c=0, \quad . \ . \ . \quad (3)$$

must have a common root.

Subtracting $x\times(2)$ from (1), we have

$$bx^3+3cx^2+3dx+e=0. \quad . \ . \ . \ . \ . \ . \quad (4)$$

Subtracting $x\times(3)$ from (2), and $x\times(5)$ from (4), we have

$$bx^2+2cx+d=0, \quad . \ . \ . \ . \ . \ . \ . \ . \quad (5)$$
$$cx^2+2dx+e=0. \quad . \ . \ . \ . \ . \ . \ . \ . \quad (6)$$

Eliminating x from (3), (5), and (6), we have

$$\begin{vmatrix} a & b & c \\ b & c & d \\ c & d & e \end{vmatrix} = 0 = ace + 2bcd - ad^2 - eb^2 - c^3,$$

the additional condition required. Now since D vanishes, it follows that, when there are three equal roots,

$$ace + 2bcd - ad^2 - eb^2 - c^3 \text{ and } ae - 4bd + 3c^2$$

must each be zero.

[See also Todhunter's Theory of Equations, §§ 187, 255.]

7. *Prove Newton's formula for finding the sums of the powers of the roots of an equation of the* nth *degree. Find the sum of the fifth powers of the roots of the equation*

$$x^4 - 7x^2 + 4x - 3.$$

[Todhunter's Theory of Equations, § 244.]

In the given example, introducing a new root $x=0$, which will not affect our result, we have to find the fifth powers of the roots of $x^5 - 7x^3 + 4x^2 - 3x = 0$. We find successively

$$s_1 = 0,\ s_2 - 14 = 0,\ s_3 + 12 = 0,\ s_4 - 7s_2 - 12 = 0,$$
$$s_5 - 7s_3 + 4s_2 = 0;\ \text{whence } s_5 = -140.$$

8. *Find the sum of the series*

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \&c.,$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \frac{1}{7^4} + \&c.,$$

each taken to an infinite number of terms.

We have by two known results in Trigonometry,

$$\cos\frac{\pi x}{2}=(1-x^2)\left(1-\frac{x^2}{3^2}\right)\left(1-\frac{x^2}{5^2}\right)\ldots.$$

$$=1-\frac{\pi^2x^2}{8}+\frac{\pi^4x^4}{384}-\ldots;$$

and we may suppose x less than unity.

Taking logarithms of both sides,

$$\log(1-x^2)+\log\left(1-\frac{x^2}{3^2}\right)+\log\left(1-\frac{x^2}{5^2}\right)+\ldots$$

$$=\log\left(1-\frac{\pi^2x^2}{8}+\frac{\pi^4x^4}{384}-\ldots\right).$$

Expanding the logarithms, and arranging the terms in ascending powers of x, we have

$$-x^2\left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)-\frac{x^4}{2}\left(1+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\right)-\ldots$$

$$=-\frac{\pi^2x^2}{8}-\frac{\pi^4x^4}{192}-\ldots$$

Equating coefficients, we have

$$1+\frac{1}{3^2}+\frac{1}{5^2}+\&c.=\frac{\pi^2}{8},$$

$$1+\frac{1}{3^4}+\frac{1}{5^4}+\&c.=\frac{\pi^4}{96}.$$

9. *In a right-angled spherical triangle prove the equations*

$$\cos c=\cos a\cos b,$$

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(c-b)}{\sin(c+b)}}.$$

The first formula is one of Napier's analogies.

For the second we have, from Napier's analogies,

$$\cos A = \tan b \cot c$$
$$= \frac{\sin b \cos c}{\cos b \sin c}.$$

Hence $$\frac{1-\cos A}{1+\cos A} = \tan^2 \frac{A}{2} = \frac{\sin c \cos b - \sin b \cos c}{\sin c \cos b + \sin b \cos c}$$
$$= \frac{\sin (c-b)}{\sin (c+b)};$$

therefore $$\tan \frac{A}{2} = \sqrt{\frac{\sin (c-b)}{\sin (c+b)}}.$$

10. *If α, β, γ, δ be the sides, and λ, λ' the diagonals of a spherical quadrilateral inscribed in a lesser circle, prove the relation*

$$\sin \frac{\alpha}{2} \sin \frac{\gamma}{2} + \sin \frac{\beta}{2} \sin \frac{\delta}{2} = \sin \frac{\lambda}{2} \sin \frac{\lambda'}{2}.$$

Let 2A, 2B, 2C, 2D be the angles subtended by α, β, γ, δ at the centre of the small circle of radius ρ; then $2A+2B$, $2B+2C$ are a pair of angles subtended by the diagonals.

Now $$A+B+C+D = \pi,$$

and $$\sin \frac{\alpha}{2} = \sin A \sin \rho,$$

$$\sin \frac{\beta}{2} = \sin B \sin \rho,$$

&c.,

$$\sin \frac{\lambda}{2} = \sin (A+B) \sin \rho,$$

&c.

Hence $\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}+\sin\frac{\beta}{2}\sin\frac{\delta}{2}=\sin^2\rho(\sin A\sin C+\sin B\sin D)$

$=\frac{1}{2}\sin^2\rho[\cos(A-C)-\cos(A+C)+\cos(B-D)-\cos(B+D)]$

$=\sin^2\rho.\cos\frac{A+B-C-D}{2}.\cos\frac{A-B-C+D}{2}$;

for $\cos\overline{A+C}+\cos\overline{B+D}=0$,

$=\sin^2\rho.\cos\left(\frac{\pi}{2}-\overline{C+D}\right)\cos\left(\frac{\pi}{2}-\overline{B+C}\right)$

$=\sin^2\rho.\cos(C+D)\sin(B+C)$

$=\sin^2\rho.\sin(A+B)\sin(B+C)$

$=\sin\frac{\lambda}{2}\sin\frac{\lambda'}{2}$.

11. *If* A′, B′, C′ *be respectively the points in which the circle inscribed to the spherical triangle* ABC *touches the sides, prove that the great circles* AA′, BB′, CC′ *pass through a common point.*

[See Todhunter's Spherical Trigonometry, §§ 139, 189.]

12. *Find the value of*

$$\frac{\alpha^2}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^2}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\gamma^2}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}.$$

The sum of these fractions is

$$\alpha^2(\beta-\gamma)(\beta-\delta)(\gamma-\delta)+\beta^2(\gamma-\alpha)(\gamma-\delta)(\alpha-\delta)+\gamma^2(\alpha-\beta)(\alpha-\delta)(\beta-\delta)\div(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta).$$

Now the dividend, which is of the fifth degree, and symmetrical in α, β, γ, vanishes when $\alpha=\beta$, or $\beta=\gamma$, or $\gamma=\alpha$, or $\delta=0$.

Hence

$$\alpha^2(\beta-\gamma)(\beta-\delta)(\gamma-\delta)+..+..$$
$$=(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)\delta\{l(\alpha+\beta+\gamma)+m\delta\}.$$

Comparing the coefficients of $\alpha^2\beta\delta$ and $\alpha^2\beta\delta^2$, we find that $l=0$, and $m=1$; and the whole expression becomes

$$\frac{(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma)\delta^2}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)},$$

or

$$\frac{\delta^2}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)},$$

which last is the simplest form required.

Afternoon, 3 *to* 6.

DIFFERENTIAL CALCULUS.

1. *If u be a function of the differences of the variables $x_1, x_2, \ldots x_n$, prove that*

$$\frac{du}{dx_1}+\frac{du}{dx_2}+\ldots.+\frac{du}{dx_n}=0.$$

It is clear that u is unaltered by increasing each of the variables by h; for this does not alter any difference. Hence

$$u=\phi(x_1, x_2, \ldots x_n)=\phi(x_1+h, x_2+h, \ldots x_n+h).$$

Expanding the last expression by Taylor's Theorem, we have

$$u=u+h\left(\frac{du}{dx_1}+\frac{du}{dx_2}+\ldots.+\frac{du}{dx_n}\right)$$
$$+\frac{h^2}{1.2}\left(\frac{d^2u}{dx_1^2}+\frac{2d^2u}{dx_1\,dx_2}+\ldots.\right)+\ldots.,$$

and each coefficient must vanish identically. Thus

$$\frac{du}{dx_1}+\frac{du}{dx_2}+\ldots\ldots+\frac{du}{dx_n}=0.$$

2. *Find the values of*

$$\frac{1}{x^2}-\frac{1}{x\tan x} \quad \textit{and} \quad x^n(\log x)^m$$

when $x=0$.

We have

$$\frac{1}{x^2}-\frac{1}{x\tan x}=\frac{\tan x-x}{x^2\tan x},$$

which is of the form $0\div 0$ when x is zero; hence, differentiating numerator and denominator, its limit is the limit of

$$\frac{\sec^2 x-1}{2x\tan x+x^2\sec^2 x}, \text{ i.e. of } \frac{\sin^2 x}{x\sin 2x+x^2},$$

which, by elementary work, ultimately becomes

$$\frac{x^2+\ldots.}{3x^2+\ldots.}, \text{ or } \frac{1}{3} \text{ when } x=0.$$

In the second example we require the limit of $\left(\frac{n}{x^m}\log x\right)^m$ when $x=0$.

Various cases may arise according to the signs of m and n.

Case 1.—m and n both positive.

The limit is then found to be 0^m or 0, the usual method being applied to the expression in the bracket, which is of the form $-\infty\times 0$.

Case 2.—m and n negative.

In this case we have 0^m also or ∞.

CASE 3. m positive and n negative.

Here the expression in the bracket becomes of the form $\infty \times -\infty$; and we have $(-\infty)^m$ or $\pm\infty$, unless m is of the form $\frac{p}{2q}$, when the limit is imaginary; for clearly $(\log x)^m$ cannot then have a real value when $x<1$.

CASE 4.—m negative and n positive.

Here the limit is the reciprocal of that in case 3, or zero.

3. *Find the maximum and minimum sections of a right cone, which stands on a given circular base; and point out the different cases which may arise.*

[This problem is completely solved in Williamson's Differential Calculus, § 148.]

4. *Find the condition among its coefficients that the three asymptotes of a curve of the third degree should pass through a common point.*

What additional condition is requisite that the curve should have a centre?

Let the general equation to a cubic be taken in the form

$$\begin{aligned}\phi(x,y) = a_1x^3 + 3b_1x^2y + 3c_1xy^2 + d_1y^3 \\ + a_2x^2 + 2b_2xy + c_2y^2 \\ + a_3x + b_3y + c_3 = 0.\end{aligned}$$

Now, if the three asymptotes meet in a point, it will be possible, by transforming the equation to this point as origin, to reduce it to the form

$$(l_1x + m_1y)(l_2x + m_2y)(l_3x + m_3y) + px + qy + r = 0,$$

where the three factors on the left are the asymptotes, and $px+qy+r=0$ the line on which lie the other points where they meet the curve [Williamson's Differential Calculus, § 204.]

But, in order that this transformation may be possible,

terms of the second degree must disappear; or, if h, k be the coordinates of the new origin, there must be no terms of the second degree in $\phi(x+h, y+k)$.

Expanding $\phi(x+h, y+k)$, and equating the coefficients of $x^2, 2xy, y^2$ to zero, we find that h, k must simultaneously satisfy the equations

$$\left.\begin{aligned} 3a_1h+3b_1k+a_2&=0,\\ 3b_1h+3c_1k+b_2&=0,\\ 3c_1h+3d_1k+c_2&=0. \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (1)$$

Hence

$$\begin{vmatrix} a_1 & b_1 & a_2 \\ b_1 & c_1 & b_2 \\ c_1 & d_1 & c_2 \end{vmatrix} = 0$$

is the condition required.

Again, if the curve has a centre, the curve when transferred to it as origin must have only terms of the third and first degree [Salmon's Higher Plane Curves, § 131], and thus the constant term must also vanish.

Thus, by proceeding as before, we see that if h, k *now represent the coordinates of the centre*, the equations (1) must be satisfied, and, in addition, we shall have

$$\phi(h, k)=0. \quad \ldots\ldots\ldots \quad (2)$$

We may observe that, in this case, the centre and the intersection of the asymptotes coincide.

5. *Eliminate, by differentiation, the arbitrary functions from the equations*

$$x-\alpha=(z-\gamma)\phi\left(\frac{y-\beta}{z-\gamma}\right),$$

$$z=\phi\left(\frac{y}{x}\right)\cos(ax+by)+\psi\left(\frac{y}{x}\right)\sin(ax+by).$$

The first equation may be written

$$\frac{x-\alpha}{y-\beta}=\frac{z-\gamma}{y-\beta}\phi\left(\frac{y-\beta}{z-\gamma}\right)=\psi\left(\frac{y-\beta}{z-\gamma}\right);$$

hence

$$z-\gamma=(y-\beta)f\left(\frac{x-\alpha}{y-\beta}\right).$$

Differentiating successively with regard to x, y, and using obvious abbreviations, we have

$$p=f',$$

$$q=f-\frac{x-\alpha}{y-\beta}\cdot f'.$$

Hence

$$p(x-\alpha)+q(y-\beta)=(y-\beta)f=z-\gamma.$$

the result of the elimination required.

In the second example change the variables by means of the conditions

$$y=x\xi, \quad ax+by=\eta.$$

We have

$$z=\phi(\xi)\cos\eta+\psi(\xi)\sin\eta,$$

and

$$\frac{d^2z}{d\eta^2}+z=0.$$

Now

$$\frac{dz}{d\eta}=p\frac{dx}{d\eta}+q\frac{dy}{d\eta},$$

and

$$x=\frac{\eta}{a+b\xi}, \quad y=\frac{\eta\xi}{a+b\xi};$$

hence

$$\frac{dz}{d\eta}=\frac{p+q\xi}{a+b\xi},$$

and, similarly,

$$\frac{d^2z}{d\eta^2}=\frac{r+s\xi}{(a+b\xi)^2}+\frac{(s+t\xi)\xi}{(a+b\xi)^2}.$$

Now, changing back to x, y, we get

$$x^2r+2xys+y^2t+(ax+by)^2z=0.$$

6. *Find the value of* $\frac{d^2u}{dx^2}+\frac{d^2u}{dy^2}+\frac{d^2u}{dz^2}$ *when*

$$u=\sqrt{x^2+y^2+z^2}+\frac{1}{\sqrt{x^2+y^2+z^2}}.$$

The result is $$\frac{2}{\sqrt{x^2+y^2+z^2}}.$$

7. *If* p *and* ω *be the pedal coordinates of any point on a plane curve, prove that the radius of curvature at that point is equal to* $p+\frac{d^2p}{d\omega^2}$. *Apply this formula to find the radius of curvature at any point on a parabola.*

[Williamson's Differential Calculus, §§ 192, 236.]

In the case of the parabola $y^2=4ax$, we have the equation in pedal coordinates,

$$p=a\sec\omega,$$

whence $$\rho=2a\sec^3\omega.$$

8. *Show how the equation to the evolute of a given curve may be determined; and find the equation to the evolute of an ellipse.*

[Williamson's Differential Calculus, §§ 237, 238, 219.]

The coordinates of the centre of curvature at the point on an ellipse whose excentric angle is ϕ are given by

$$xa=(a^2-b^2)\cos^3\phi,\ yb=-(a^2-b^2)\sin^3\phi.$$

Hence $$(xa)^{\frac{2}{3}}+(yb)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$$

is the equation to the evolute.

9. *If a plane move in such a manner that two curves in it always touch two fixed plane curves, show how to construct the*

centre of curvature of the curve described by any point in the moving plane, for any position.

[Williamson's Differential Calculus, § 297.]

10. *Find the form, near the origin, of the curve*

$$y^4+axy^2=2ax^3+x^4,$$

and determine the equations of its asymptotes.

Solving for y we have

$$2y^2=-ax\pm x\sqrt{a^2+8ax+4x^2}.$$

First let x be small, then y is also small; expanding in ascending powers of x and neglecting powers above the third, we have

$$2y^2=-ax\pm\left(ax+4x^2-\frac{6x^3}{a}\right);$$

$$\therefore\quad y^2=2x^2-\frac{3x^3}{a},\text{ or }-ax-2x^2.$$

Whence $\quad y=\pm\sqrt{2}\left(x-\frac{3x^2}{4a}\right),$ or $y^2+ax=0.$

These equations give us the shape of the curve near the origin, viz.:—two parabolæ touching the lines $y^2-2x^2=0$ at the origin, and lying beneath them, with axes vertical; and a parabola with vertex at the origin, and axis in the negative direction of the axis of x.

The tracing of the whole curve presents no difficulty.

Next let x be large; then, expanding in descending powers of x,

$$2y^2=-ax\pm(2x^2+2ax+\text{lower powers}),$$

and we must take the upper sign only;

$$\therefore\quad y^2=x^2+\tfrac{1}{2}ax+\ldots$$

and $$y=\pm(x+\tfrac{1}{4}a+\ldots).$$

Thus the asymptotes are

$$4(x\pm y)+a=0.$$

11. *Prove that the equation*

$$r^2-2(a+b\cos\theta)r+c^2=0$$

represents a Cartesian oval; and find when it has three real foci, and when but one.

If P be a point on a Cartesian oval of which the origin O is one focus, and a point C at a distance d along the initial line another, the equation to the oval may be written

$$l\,.\,\mathrm{OP}+m\,.\,\mathrm{CP}=d;$$

or, in polar coordinates,

$$r^2(l^2-m^2)-2rd(l-m^2\cos\theta)+d^2(1-m^2)=0.$$

Now the given equation is of a similar form, and therefore represents a Cartesian oval. To determine the constants l, m, d we have, on comparing coefficients,

$$\frac{l^2-m^2}{1}=\frac{dl}{a}=-\frac{dm^2}{b}=\frac{d^2(1-m^2)}{c^2};$$

whence $lb=-m^2a$, and, substituting for l,

$$a^2m^2=b^2-bd,\quad (bd-c^2)m^2=bd;$$

and therefore

$$d^2b+d(a^2-b^2-c^2)+bc^2=0.$$

This equation has real roots if

$$(a^2-b^2-c^2)^2>4b^2c^2;$$

i. e. $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2>0$;

i. e. $(a+b+c)(b+c-a)(c+a-b)(a+b-c)<0.$

Thus there are three real foci if one of the quantities a, b, c is greater than the sum of the other two, and but one if any two of them are together greater than the third.

Tuesday, August 2, 1881.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

Morning, 10 *to* 1.

INTEGRAL CALCULUS.

1. *Find the values of the integrals*

$$\int \frac{dx}{(1+x)^3(2+x)^4}, \quad \int \{x+(1+x^2)^{\frac{1}{2}}\}^{\frac{1}{2}} dx, \quad \int \frac{dx}{1-\cos^4 x}.$$

In the first integral let $1+x=\frac{1}{y}$, it reduces to

$$-\int \frac{y^5\,dy}{(1+y)^4}.$$

Now let $1+y=t$, and it becomes

$$\int t^{-4}(1-t)^5\,dt.$$

Expanding and integrating, we have

$$-\frac{1}{3t^3}+\frac{5}{2t^2}-\frac{10}{t}-10\log t+5t-\frac{t^2}{2},$$

where $$t=1+y=\frac{2+x}{1+x}.$$

In the second integral let

$$x+(1+x^2)^{\frac{1}{2}}=y, \quad \text{and} \quad -x+(1+x^2)^{\frac{1}{2}}=\frac{1}{y};$$

then $$2x=\frac{y^2-1}{y}, \quad 2dx=\frac{y^2+1}{y^2}\,dy.$$

Thus the integral becomes

$$\tfrac{1}{2}\int y^{\frac{1}{2}}\frac{(y^2+1)}{y^2}\,dy=\tfrac{2}{5}y^{\frac{5}{2}}-\tfrac{2}{3}y^{-\frac{3}{2}},$$

where y is known in terms of x.

The third integral is

$$\tfrac{1}{2}\int\frac{dx}{1-\cos^2 x}+\tfrac{1}{2}\int\frac{dx}{1+\cos^2 x}$$

$$=\tfrac{1}{2}\int\operatorname{cosec}^2 x\,dx+\tfrac{1}{2}\int\frac{\sec^2 x\,dx}{2+\tan^2 x}$$

$$=-\frac{\cot x}{2}+\frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right).$$

2. *Evaluate the definite integrals*

$$\int_0^\infty\frac{dx}{(x^2+\alpha^2)(x^2+\beta^2)},\quad\int_\beta^\alpha\frac{dx}{x}\sqrt{\frac{\alpha^2-x^2}{x^2-\beta^2}}.$$

We have

$$\frac{1}{(x^2+\alpha^2)(x^2+\beta^2)}=\frac{1}{\alpha^2-\beta^2}\left\{\frac{1}{x^2+\beta^2}-\frac{1}{x^2+\alpha^2}\right\}.$$

The first integral therefore becomes

$$\frac{1}{\alpha^2-\beta^2}\left\{\int_0^\infty\frac{dx}{x^2+\beta^2}-\int_0^\infty\frac{dx}{x^2+\alpha^2}\right\}$$

$$=\frac{\pi}{2(\alpha^2-\beta^2)}\left(\frac{1}{\beta}-\frac{1}{\alpha}\right)=\frac{\pi}{2\alpha\beta(\alpha+\beta)}.$$

In the second integral let

$$x^2=\alpha^2\sin^2\theta+\beta^2\cos^2\theta;$$

it reduces to

$$\int_0^{\frac{\pi}{2}}\frac{(\alpha^2-\beta^2)\cos^2\theta\,d\theta}{\alpha^2\sin^2\theta+\beta^2\cos^2\theta}$$

$$= \int_0^{\frac{\pi}{2}} \frac{a^2 \, d\theta}{a^2 \sin^2\theta + \beta^2 \cos^2\theta} - \int_0^{\frac{\pi}{2}} d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{a^2 \sec^2\theta \, d\theta}{a^2 \tan^2\theta + \beta^2} - \frac{\pi}{2}$$

$$= \left(\frac{a}{\beta} - 1\right)\frac{\pi}{2} = \frac{(a-\beta)\pi}{2\beta}.$$

3. *Prove the relation*

$$\int_0^{\frac{\pi}{2}} \frac{dx}{(1+\cos a \cos x)^{\frac{3}{2}}} = \frac{1}{\sin^2 a} \int_0^{\frac{\pi}{2}} dx (1-\cos a \cos x)^{\frac{1}{2}}.$$

A more general case of this theorem is proved in Williamson's Integral Calculus, § 75.

4. *Prove that the length of any arc of a hyperbola may be expressed in terms of a right line and of the arcs of two ellipses.*

[See Williamson's Integral Calculus, §§ 159, 160.]

5. *Determine the area of the pedal of an ellipse taken with respect to any internal origin.*

[See Williamson's Integral Calculus, § 143.]

6. *Trace the curve*

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{l^2} + \frac{y^2}{m^2},$$

and find its entire area.

On projecting this curve by means of the transformation $\frac{x}{a}=\frac{X}{c}$, $\frac{y}{b}=\frac{Y}{c}$, we have

$$(X^2+Y^2)^2=\frac{c^2a^2X^2}{l^2}+\frac{c^2b^2Y^2}{m^2},$$

or, in polar coordinates,

$$r^2=\frac{c^2a^2\cos^2\theta}{l^2}+\frac{c^2b^2\sin^2\theta}{m^2}.$$

The projection is therefore a closed oval curve (the *inverse* of an ellipse); and the original curve is of a somewhat similar shape. In fact, if we take the ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{l^2}+\frac{y^2}{m^2}=1,$$

and draw through their centre a line meeting them in Q, R, and then take P on this line so that

$$OP.OR=OQ^2,$$

the locus of P is the given curve.

For the area see Williamson, § 141.

7. *Find the length of the arc, and also the area, cut off from the curve* $y^3=ax^2$ *by the line* $y=mx$.

The length of the arc is easily found to be

$$\int\sqrt{\left\{\frac{4a+9y}{4a}\right\}}\,dy,$$

the limits being $\frac{a}{m^2}$ and 0.

This becomes

$$\frac{a\{(4m^2+9)^{\frac{3}{2}}-8m^3\}}{27m^3}.$$

Again, the area is given by

$$\int (a^{\frac{1}{3}}x^{\frac{2}{3}} - mx)dx$$

between similar limits. This reduces to

$$\frac{a^3}{10m^5}.$$

8. *Express in gamma-functions the area of a loop of the curve* $r^n = a^n \cos n\theta$, *and also its length.*

The area of the loop is $\int r^2 d\theta$, the limits being $\frac{\pi}{2n}$ and 0; this, on writing $n\theta = \phi$, becomes

$$\frac{1}{n}\int a^2 (\cos\phi)^{\frac{2}{n}} d\phi,$$

the limits being $\frac{\pi}{2}$ and 0; and, by Williamson's Integral Calculus, § 122 (62), this is

$$\frac{a^2\sqrt{\pi}}{2n} \cdot \frac{\Gamma\left(\frac{1}{2}+\frac{1}{n}\right)}{\Gamma\left(1+\frac{1}{n}\right)}.$$

The expression for the length is worked out in Williamson, § 156, Ex. 3: it is

$$\frac{\sqrt{\pi}\Gamma\left(\frac{1}{2n}\right)}{n\Gamma\left(\frac{n+1}{2n}\right)}.$$

9. *If the segment of a parabola cut off by any focal chord make a complete revolution round the chord, find the volume of the generated surface in terms of the length of the chord and the parameter of the parabola.*

This may be worked by Guldinus' properties [Williamson, § 178], or as follows:—Let AB be a chord of length l, O its middle point, OP its diameter, QVQ' any double ordinate, $PV=x$, $QV=y$, semi-latus rectum of curve $=a$, $\angle POA=\theta$; then, drawing an ordinate RR' indefinitely near to QQ', and considering the volume generated by the revolution of the figure QQ'R'R about AB, we have

$$dV=4\pi y VO \sin^2\theta dx.$$

Now $VO=PO-x=\frac{1}{4}l-x$, $l\sin^2\theta=2a$, $y^2=lx$; hence

$$V=\int_0^{\frac{1}{4}l} 2\pi a(l-4x)\sqrt{\frac{x}{l}}.dx=\frac{\pi a l^2}{15}.$$

10. *Show how to divide the quadrantal arc of an ellipse into two parts whose difference shall be equal to the difference between the semiaxes.*

A method depending on the Integral Calculus is given in Williamson, § 158, Ex. 2. A better one, however, is in Salmon's Conic Sections, § 400.

Afternoon, 3 *to* 6.

GEOMETRY.

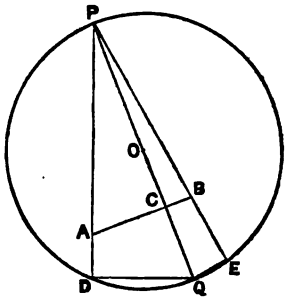
1. *Inscribe a square in a given triangle.*

[This problem is worked in the Appendix to Todhunter's Euclid.]

2. A *and* B *are two fixed points within a given circle. Describe a circle passing through* A *and* B, *and intersecting the circle in* D *and* E, *so that the chords* DA, EB *meet on the given circle.*

Let O be the centre of the circle; from O draw OC perpendicular to AB, meeting the circle in P, Q; join PA, PB, and produce these lines to meet the circle again in D, E.

Fig. 22.



Join QD, QE; then, because the angles at C, D are right angles, therefore the triangles PAC, PDQ are similarly, and therefore the rectangle PA . PD = the rectangle PC . PQ.

Similarly PB . PE = PC . PQ.

Hence PA . PD = PB . PE;

and therefore through ABDE can be drawn a circle, which is the circle required.

Another solution may be obtained by joining QA, QB.

3. AB, BC, CD, DA *are four equal rods*, OA *and* OC *are two rods equal to each other, the rods are hinged together at* O, A, B, C, D; *prove that, if* O *is fixed and* B *describes a circle passing through* O, D *will describe a straight line.*

From A draw AN perpendicular to BD which it bisects; then O, B, D are clearly in one straight line, and

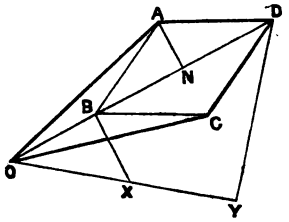
$$\text{rect. OB . OD} = \text{sq. ON} \backsim \text{sq. BN}$$
$$= \text{sq. AO} \backsim \text{sq. AB},$$

which is a constant quantity.

Let OX be the diameter of the circle through O which B describes; from D draw DY perpendicular to OX; then because $\angle$OBX is a right angle, therefore

$$\text{rect. OB . OD} = \text{rect. OX . OY}.$$

Fig. 23.



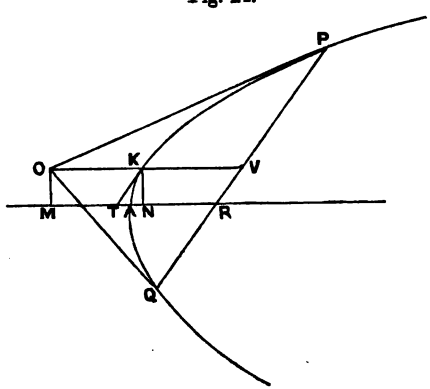
But OX is a fixed length, and the rectangle OB.OD is constant; therefore OY is constant, and the locus of D is a line at right angles to OX.

NOTE.—By this instrument we can draw the inverse of any given curve.

4. OP, OQ *are two tangents to a parabola, and* OM *is drawn perpendicular to the axis; prove that* PQ *cuts the axis in a point* R *such that* MR *is bisected by the vertex of the parabola.*

Let A be the vertex, OKV the diameter through O

Fig. 24.



meeting the curve in K and PQ in V, KT the tangent

at K, KN the ordinate at K, then we have OK=KV, MN=OK, TR=KV.

Hence $MN=TR$,

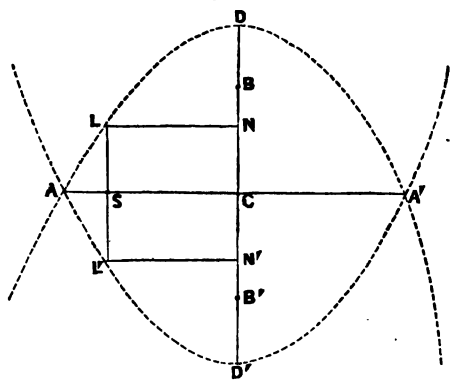
but $AN=TA$,

therefore $MA=AR$.

5. *The extremities of the latera recta of all conics which have a given major axis lie on two parabolas.*

Let AA′ be the given major axis, C the centre, DD′ the points where the auxiliary circle cuts the minor axis produced. Let LL′, S, BB′ be any simultaneous positions of the latus rectum, focus, and minor axis; from LL′ draw LN, L′N′ perpendicular to BB′.

Fig. 25.



Then $SL.AC=BC^2=AC^2-SC^2$;

therefore $SC^2=AC(AC-SL)=DC(CD-CN)$,

or $LN^2=DN.DC$;

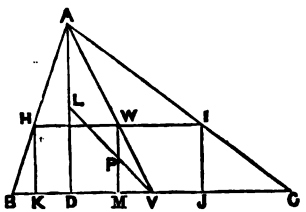
therefore the locus of L is a parabola with vertex D, axis DC, and latus rectum equal to DC or CA.

Similarly the locus of L′ is an equal parabola with vertex D′, axis D′C.

6. *Find the locus of the middle points of rectangles described in a given triangle.*

Let ABC be the triangle, and let us consider rectangles whose sides are parallel and perpendicular to BC. From A draw AD perpendicular to BC, bisect BC in V, and AD

Fig. 26.



in L, and join AV and VL; let HI be the side parallel to BC of any inscribed rectangle HIJK; then AV meets HI in its middle point W; and if WM be drawn parallel to AD, its middle point P is in the line VL; but P is the middle point of the rectangle HIJK: therefore the middle points of all rectangles inscribed in the triangle ABC, with one side parallel to BC, is a line VL passing through the middle points of BC and the perpendicular from A on BC.

7. A, B, C, D *lie upon a conic. Prove that the intersection of* AB *and* CD *is the pole of the line joining the intersections of* AC, BD *and of* AD, BC.

[Besant's Conic Sections, § 197; Salmon's Conic Sections, § 146, Ex. 1.]

8. *Find the locus of the middle points of all focal chords of an ellipse.*

Let O be any fixed point; let QQ′ be any chord through O, V its middle point; also let qq' be the corresponding chord in the auxiliary circle, v its middle point. Join v and the centre C of the ellipse; then Cv is at right angles to qq', and the locus of v is therefore a circle on OC as

diameter. Hence the locus of V is an ellipse, similar to and similarly situated with the given ellipse, with OC for a diameter. In the given example, which is a particular case of this theorem, O is the focus of the given ellipse.

9. *Find the equations to the asymptotes of the hyperbola*

$$xy+ax+by+c^2=0.$$

The given curve may be written

$$(x+b)(y+a)=ab-c^2,$$

whence it is obvious that the asymptotes are

$$x+b=0, \quad y+a=0,$$

the axes being rectangular or oblique.

10. *If from every point in the major axis of an ellipse as centre a circle be described with radius equal to the ordinate at that point, find the envelope of the circles.*

The ellipse being $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, any point P on it may be represented by the coordinates

$$x=a\cos\phi,$$
$$y=b\sin\phi;$$

and the circle whose centre is the foot of the ordinate at P, and radius equal to that ordinate, will have for equation

$$(x-a\cos\phi)^2+y^2=b^2\sin^2\phi,$$

or
$$x^2+y^2-b^2-2ax\cos\phi+(a^2+b^2)\cos^2\phi=0,$$

which involves the parameter $\cos\phi$ in the second degree. [Salmon's Conic Sections, § 283.]

The envelope will therefore be

$$(x^2+y^2-b^2)(a^2+b^2)=a^2x^2,$$

or
$$\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1,$$

which is a concentric ellipse with the same minor axes as the given ellipse.

This problem is an interesting example of an envelope which does not touch all the curves of the system. See Todhunter's Differential Calculus, § 338.

11. *Given two triangles, both self-conjugate with regard to a conic; prove that their six vertices lie in a conic.*

Let ABC, DEF be the two triangles, and let BC meet DE, DF in P, Q; join AE, AF. Then Q is the pole of AE, and P of AF. Now the anharmonic ratio of the pencil $\{A . EBCF\}$ = the anharmonic ratio of the range formed by their poles,

$$= \{QCBP\} = \{D . QCBP\} = \{D . FCBE\} = \{D . EBCF\}.$$

Hence, if we describe a conic through the five points A, B, C, E, F, it will also pass through D, since the anharmonic ratio of the pencil formed by joining D to the four points EBCF is the same as that formed by joining A to the same points.

12. *Prove that confocal conics cut at right angles.*

[Salmon's Conic Sections, § 188.]

SECOND B.A. AND B.Sc. HONOURS EXAMINATION.

Monday, November 8, 1880.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

Morning, 10 *to* 1.

ALGEBRA AND TRIGONOMETRY.

1. *If all the coefficients in the equation*

$$x^n+p_1x^{n-1}+p_2x^{n-1}+\&c.+p_n=f(x)=0$$

be whole numbers, and if $f(0)$ *and* $f(1)$ *be each odd integers, prove that the equation cannot have a commensurable root.*

The equation cannot have a fractional root, for all the coefficients are integers, and that of x^n is 1; it cannot have an even root, for $f(0)$ or p_n is odd, and hence $f(2m)$ will be odd, since all the terms but the last are even. It cannot have an odd root; for if x is odd,

x^n = an odd number

= an even number + 1.

Hence

$f(x)$ = an even number + $1+p_1+p_2+\ldots+p_n$

= an even number + $f(1)$

= an odd number,

and therefore cannot vanish.

Hence the given equation cannot have any commensurable root.

2. *Find the values of* $x_1, x_2 \ldots x_n$ *which satisfy the following system of simultaneous equations:—*

$$\frac{x_1}{a_1-b_1}+\frac{x_2}{a_1-b_2}+\ldots.+\frac{x_n}{a_1-b_n}=1,$$

$$\frac{x_1}{a_2-b_1}+\frac{x_2}{a_2-b_2}+\ldots.+\frac{x_n}{a_2-b_n}=1,$$

$$\ldots\ldots\ldots$$

$$\frac{x_1}{a_n-b_1}+\frac{x_2}{a_n-b_2}+\ldots.+\frac{x^n}{a_n-b_n}=1.$$

Let us, as is always possible, split up

$$\frac{(x-a_2)(x-a_3)\ldots.(x-a_n)}{(x-b_1)(x-b_2)\ldots.(x-b_n)}, \quad \text{or} \quad \phi(x),$$

into partial fractions, viz.:—

$$\frac{\lambda_1}{x-b_1}+\frac{\lambda_2}{x-b_2}+\ldots.+\frac{\lambda_n}{x-b_n}.$$

Multiply the given equations successively by $\lambda_1, \lambda_2 \ldots. \lambda_n$, and add; we get

$$x_1\left\{\frac{\lambda_1}{a_1-b_1}+\frac{\lambda_2}{a_1-b_2}+..+\frac{\lambda_n}{a_1-b_n}\right\}$$

$$+x_2\left\{\frac{\lambda_1}{a-b_1}+\frac{\lambda_2}{a_2-b_2}+..+\frac{\lambda_n}{a_2-b_n}\right\}+\ldots.$$

$$=\lambda_1+\lambda_2+..+\lambda_n. \quad \ldots\ldots\ldots \quad (A)$$

But since

$$\frac{\lambda_1}{x-b_1}+\ldots.+\frac{\lambda_n}{x-b_n}=\phi(x),$$

$$\therefore \quad \frac{\lambda_1}{a_r-b_1}+\ldots.+\frac{\lambda_n}{a_r-b_n}=\phi(a_r)=0,$$

except when $r=1$. Hence (A) becomes

$$x_1\phi(a_1)=\lambda_1+\lambda_2+\ldots.+\lambda_n.$$

To find $\lambda_1+\lambda_2+\ldots.+\lambda_n$, equate the coefficients of x^{n-1} in the identity

$$\frac{(x-a_2)\ldots.(x-a_n)}{(x-b_1)\ldots.(x-b_n)}=\frac{\lambda_1}{x-b_1}+\ldots.+\frac{\lambda_n}{x-b_n}$$

when cleared of fractions.

We get

$$\lambda_1+\lambda_2+\ldots.+\lambda_n=1.$$

Hence

$$x_1=\frac{1}{\phi(a_1)}=\frac{(a_1-b_1)\ldots.(a_1-b_n)}{(a_1-a_2)\ldots.(a_1-a_n)};$$

similarly for x_2, x_3, &c.

3. *Find the true value of*

$$\sqrt{x^2+ax}-\sqrt{x^2+bx} \quad \textit{when } x=\infty.$$

We may work this by the Binomial Theorem, or as follows:—

We have

$$\sqrt{x^2+ax}-\sqrt{x^2+bx}=\frac{(x^2+ax)-(x^2+bx)}{\sqrt{(x^2+ax)}+\sqrt{(x^2+bx)}}$$

$$=\frac{(a-b)x}{x+..+x+..}=\frac{a-b}{2}.$$

4. *Define an invariant of a function, and find the greatest number of independent invariants of a binary quantic of the* nth *degree.*

This question is discussed at length in Lesson xii. in Salmon's Higher Algebra, more particularly in §§ 119, 121, 122.

5. *Being given the system of equations,*

$$a_1 = x_1,$$
$$a_2 = 2bx_1 + x_2,$$
$$a_3 = 3b^2x_1 + 3bx_2 + x_3,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$a_n = n\,b^{n-1}x_1 + \frac{n(n-1)}{1.2}b^{n-2}x_2 + \ldots + x_n,$$

find the value of x_n *in terms of* $a_1, a_2, \ldots a_n$, *&* b.

Multiply the last equation by 1, the last but one by nb, the last but two by $\frac{n(n-1)b^2}{1.2}$, and so on; add and subtract the results alternately, and we find that the coefficient of $b^{n-r}x_r$ on the right hand is

$$\frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots.r}\left\{1-(n-r)+\frac{(n-r)(n-r-1)}{1.2} \ldots\ldots \pm 1\right\}$$
$$= \frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots.r}(1-1)^{n-r}$$

$= 0$, except when $r = n$. In this case the coefficient is unity, and we have

$$x_n = a_n - n\,a_{n-1}b + \frac{n(n-1)}{1.2}a_{n-2}b^2 - \ldots + (-1)^{n-1}n a_1 b^{n-1}.$$

6. *Prove the identical equation*

$$\frac{1}{\pi^2 - x^2} + \frac{1}{4\pi^2 - x^2} + \frac{1}{9\pi^2 - x^2} + \&c. = \frac{1}{2}\left(\frac{1}{x^2} - \frac{1}{x \tan x}\right).$$

We have

$$\sin x = x\left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right), \ldots \quad \ldots \quad (1)$$

$$\sin(x+y)=(x+y)\left\{1-\frac{(x+y)^2}{\pi^2}\right\}\left\{1-\frac{(x+y)^2}{4\pi^2}\right\}..$$

$$=x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)..+y\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)..$$

$$-\frac{2x^2y}{\pi^2}\left(1-\frac{x^2}{4\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)..-\frac{2x^2y}{4\pi^2}\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{9\pi^2}\right)..$$

$$-..\text{ \&c.}-\text{higher powers of } y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Now dividing (2) by (1), we have

$$\frac{\sin(x+y)}{\sin x}=1+y\left\{\frac{1}{x}-\frac{2x}{\pi^2-x^2}-\frac{2x}{4\pi^2-x^2}-..\right\}$$

$$+\text{higher powers of } y. \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But also $$\frac{\sin(x+y)}{\sin x}=\cos y+\sin y\cot x$$

$$=1+y\cot x-\tfrac{1}{2}y^2-. \quad . \quad . \quad (4)$$

Hence, equating the coefficients of y in (3) and (4), the result required is readily deduced.

NOTE. The same result is at once obtained by differentiating the logarithms of the expressions on both sides of (1).

7. *Eliminate θ between the equations*

$$\frac{x}{a}\cos\theta-\frac{y}{b}\sin\theta=\cos 2\theta,$$

$$\frac{x}{a}\sin\theta+\frac{y}{b}\cos\theta=2\sin 2\theta.$$

Solving for $\frac{x}{a}$, $\frac{y}{b}$, we at once obtain

$$\frac{x}{a}=\cos\theta\cos 2\theta+2\sin\theta\sin 2\theta$$

$$=\cos\theta(1+2\sin^2\theta),$$

$$\frac{y}{b}=2\cos\theta\sin 2\theta-\sin\theta\cos 2\theta$$

$$=\sin\theta(1+2\cos^2\theta);$$

$$\therefore \quad \frac{x}{a}+\frac{y}{b}=(\sin\theta+\cos\theta)^2=(1+\sin 2\theta)^{\frac{1}{2}},$$

$$\frac{x}{a}-\frac{y}{b}=(\cos\theta-\sin\theta)^2=(1-\sin 2\theta)^{\frac{1}{2}};$$

$$\therefore \quad \left(\frac{x}{a}+\frac{y}{b}\right)^{\frac{1}{2}}+\left(\frac{x}{a}-\frac{y}{b}\right)^{\frac{1}{2}}=1+\sin 2\theta+1-\sin 2\theta=2,$$

the result of the elimination required.

8. *In a spherical triangle prove the equation*

$$\frac{\sin a}{\sin A}=\sqrt{\frac{1-\cos a\cos b\cos c}{1+\cos A\cos B\cos C}}.$$

We have $\cos a-\cos b\cos c=\cos A\sin b\sin c,$

$$\therefore \quad \cos^2 a-\cos a\cos b\cos c=\cos a\cos A\sin b\sin c,$$

and $$1-\cos a\cos b\cos c=\sin^2 a+\cos a\cos A\sin b\sin c$$
$$=n^2(\sin^2 A+\cos a\cos A\sin B\sin C),$$

where $$n=\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}.$$

Again $$\sin^2 A+\cos a\cos A\sin B\sin C$$
$$=1-\cos A(\cos A-\cos a\sin B\sin C)$$
$$=1+\cos A\cos B\cos C.$$

Thus $$\frac{1-\cos a\cos b\cos c}{1+\cos A\cos B\cos C}=n^2=\frac{\sin^2 a}{\sin^2 A},$$

from which the required result immediately follows.

9. *If through any point on a lesser circle of a sphere two arcs be drawn at right angles to each other, find the envelope of the great circle joining the points in which the arcs intersect the circle.*

Let O be a point on a small circle, OA, OB any two arcs at right angles, meeting it in A, B; let C be the centre of the circle, c its radius, $\angle COA=\beta$. Then

$$\tan \tfrac{1}{2}OA \cos \beta = \tan c = \tan \tfrac{1}{2}OB \sin \beta. \quad . \quad . \quad (1)$$

Now it is easy to see that the equation of a great circle in polar coordinates is of the form

$$\tan r \cos(\theta - a) = \tan p. \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Let this be the great circle AB; then, since it passes through A and B, we have

$$\tan OA \cos(\beta - a) = \tan OB \sin(\beta - a) = \tan p,$$

or, by (1),

$$\frac{2 \tan c \cos\beta \cos(\beta - a)}{\cos^2\beta - \tan^2 c} = \frac{2 \tan c \sin\beta \sin(\beta - a)}{\sin^2\beta - \tan^2 c} = \tan p;$$

whence, adding the numerators, we have

$$2 \tan c \cos a = \tan p \,(1 - 2\tan^2 a),$$

or $$\tan p \sec a = \text{constant, for all values of } \beta.$$

But if in (2) $\theta = 0$, we have

$$\tan r = \tan p \sec a = \text{constant};$$

hence the circle AB meets OC in a fixed point, or its envelope is a point.

We might have proceeded thus:—Project the whole figure, with the centre of the sphere as vertex, on the tangent plane at O, and let accented letters denote the projections; then the circle becomes an ellipse, OA′, OB′ two chords at right angles, and A′B′ a third straight line which, by a theorem in conics, passes through a fixed point on the normal at O. Hence the great circle AB passes through a fixed point on the arc OC.

10. *If a', b', c' denote the arcs joining the angular points* ABC *to any point on the surface of the sphere, prove the following determinant relation:—*

$$\begin{vmatrix} 1, & \cos a, & \cos b, & \cos c' \\ \cos a, & 1, & \cos c, & \cos b' \\ \cos b, & \cos c, & 1, & \cos a' \\ \cos c', & \cos b', & \cos a', & 1 \end{vmatrix} = 0,$$

where a, b, c are the sides of the spherical triangle.

Apply this equation to find the radius of a sphere circumscribed to a given tetrahedron, in terms of the lengths of the edges of the tetrahedron.

[Salmon's Solid Geometry, § 56; Todhunter's Spherical Trigonometry, § 163.]

Afternoon, 3 to 6.

PURE AND COORDINATE GEOMETRY.

1. *The base of a triangle is fixed in position; find the equation to the locus of its vertex when the vertical angle is double one of its base angles.*

Trace the locus in question, finding the position of its asymptote. Find also the area of the oval portion of the curve.

Let OB be the base, O the origin, P the vertex, $OB=c$, $OP=r$, $\angle BOP=\theta$, $\angle OPB=2\theta$.

Then $r \sin 2\theta = c \sin 3\theta$, in polar coordinates,

or $2x(x^2+y^2)=c(3x^2-y^2)$ in rectangular coordinates, are the equations of the curve required. . (1)

To trace the curve we have

$$y^2=x^2 \cdot \frac{3c-2x}{2x+c}.$$

Hence x must lie between $-\frac{1}{2}c$ and $+\frac{3}{2}c$; and therefore the asymptote must be perpendicular to OB. Its equation is clearly

$$2x+c=0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Again, the loop extends from $x=0$ to $x=\frac{3}{2}c$, and its greatest height is found, by differentiating y, to be when $x=\frac{1}{2}c\sqrt{3}$.

Its area is best found by means of polar coordinates, and is equal to $\int\frac{1}{2}r^2d\theta$, where the limits of $\tan\theta$ are those of $y:x$ at the origin, or $\pm\sqrt{3}$. Hence we require the value of

$$\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}\frac{c^2\sin^2 3\theta\,d\theta}{2\sin^2 2\theta}$$

$$=\frac{c^2}{8}\int_{-\frac{1}{3}\pi}^{\frac{1}{3}\pi}(16\cos^2\theta-8+\sec^2\theta)d\theta$$

$$=\frac{3c^2\sqrt{3}}{4}.$$

2. *If two intersecting Cartesian ovals have the same foci, prove that they cut orthogonally.*

[Williamson's Differential Calculus, page 415.]

3. *Show that the evolute of a curve is the envelope of all its normals; and find the equation of the evolute to the curve*

$$y(x^2+y^2)=ax^2.$$

[Williamson's Differential Calculus, § 237.]

In the rider let $x=ty$, then we have

$$y(1+t^2)=at^2,\quad x(1+t^2)=at^3,$$

on substituting for x in the given equation.

The normal at the point (x, y) is easily found to be

$$2Y+X(t^3+3t)=a(t^4+2t^2),\quad\ldots\quad(1)$$

where X, Y are current coordinates.

Differentiating, to find the envelope of the normal,

$$3X(t^2+1)=4at(t^2+1),$$

$$\therefore\quad 3X=4at;$$

and, on substituting in (1), we have for the equation to the evolute,

$$27X^4+288X^2a^2+512a^3Y=0.$$

4. *Find the locus of points whose polar conics with regard to a cubic curve are a parabola.*

[Salmon's Higher Plane Curves, § 187.]

5. *Find the number of distinct species of central cubics; and prove that all cubics can be got by projection from central cubics.*

Taking Cartesian coordinates, with the origin at the centre of the curve, the curve must not have any terms but those of odd degree [Salmon's Higher Plane Curves, § 131]. Its equation may then be written

$$lx+my=px^3+qx^2y+rxy^2+sy^3.$$

Changing the directions of the axes, we may reduce this to

$$y=ax^3+bx^2y+cxy^2+dy^3,$$

which is resolvable into five cases, as in Salmon, §§ 196, 39.

6. *Define a line of curvature on a surface, and prove that the intersection of two confocal surfaces of the second degree is a line of curvature for both surfaces.*

[Salmon's Solid Geometry, § 303.]

Let two confocals be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \quad \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}=1.$$

Then at any common point (x, y, z)

$$\frac{xdx}{a^2}+\frac{ydy}{b^2}+\frac{zdz}{c^2}=\frac{xdx}{a^2+\lambda}+\frac{ydy}{b^2+\lambda}+\frac{zdz}{c^2+\lambda}=0.$$

$$\therefore \quad \frac{xdx}{a^2(a^2+\lambda)(b^2-c^2)}=\frac{ydy}{\dots}=\frac{zdz}{\dots}.$$

Hence $\frac{x(b^2-c^2)}{\dots} dy\,dz + \dots + \dots = 0$;

and therefore the arc dx, dy, dz is parallel to an axis of the central section parallel to the tangent plane at P, or, in other words, is a line of curvature [Salmon, § 196].

7. *If a plane cut a constant volume from a given right cone, prove that the minor axis of the section is of constant length.*

Let C be the vertex of the cone, AB the major axis of any section of the kind referred to, S the point of contact of the inscribed circle of ABC with AB; then S is also the focus of the section. Let CD be drawn perpendicular to AB.

Then volume cut off by the plane considered is

$\frac{1}{3}$CD × (area of section) = $\frac{1}{3}\pi$. CD × (rectangle contained by semiaxes of the section)

$= \frac{1}{6}\pi$. CD . AB × (semi-axis minor)

$= \frac{1}{6}\pi\, ab \sin C$ × (semi-axis minor). . . . . (1)

Again,

(semi-axis minor)^2 = AS . SB = $(s-a)(s-b) = ab \sin^2 \frac{1}{2}C$;

$\therefore$ volume cut off becomes

$$\frac{1}{3}\pi (\text{semi-axis minor})^3 . \tan \frac{C}{2}$$

by substituting for ab in (1); and since C is a constant angle, and the volume is constant, the axis minor is also constant.

NOTE. The area of the triangle ABC is also constant.

8. *Being given a fixed section of an ellipsoid, if a variable section be drawn parallel to a given plane, find the locus of the vertex of a cone of the second degree which passes through both curves of section.*

We may project the ellipsoid into a sphere; and we

then see that the vertex of the cone lies in the plane through the centre of the sphere perpendicular to both the given planes or to their common section. We have, therefore, only to consider the section of the sphere by this plane; and our problem reduces to this:—"Find the locus of the intersection of the lines joining the extremities of a fixed chord in a circle and another chord parallel to a fixed straight line."

Let AB be the fixed chord, PQ any position of the variable chord, CD the diameter of the circle which bisects PQ; then if AP, BQ meet in X, we have CP=CQ, and therefore $\angle CAP = \angle CQB$. Hence CX subtends equal angles at A and B; therefore [Besant's Conic Sections, § 137], if a rectangular hyperbola be described through C with AB as diameter, it will pass through X; hence the locus of X is this hyperbola, which also passes through D.

Reverting now to our original problem, we find that the locus of the vertex of the cone is a hyperbola in the plane conjugate to the intersection of the given planes, with the intersection of this plane and the fixed plane as diameter, and passing through the extremities of that diameter of the ellipsoid which is conjugate to the moving plane.

9. *Find expressions for the radius of curvature and of torsion of a curve in space whose equations are of the form*

$$x=\phi(t), \quad y=\psi(t), \quad z=\chi(t).$$

The results may be easily deduced from those in Salmon's Solid Geometry, §§ 397, 399.

10. *Find the differential equation of surfaces of revolution; and determine the conditions in order that*

$$ax^3+by^3+cz^3+dxyz=\text{constant}$$

should represent a surface of revolution.

In a surface of revolution, F=0, the normal

$$\frac{X-x}{\frac{dF}{dx}}=\frac{Y-y}{\frac{dF}{dy}}=\frac{Z-z}{\frac{dF}{dz}}$$

always intersects a fixed straight line

$$\frac{X-f}{l}=\frac{Y-g}{m}=\frac{Z-h}{n}.$$

But the condition for this is

$$\begin{vmatrix} x-f & y-g & z-h \\ l & m & n \\ \frac{dF}{dx} & \frac{dF}{dy} & \frac{dF}{dz} \end{vmatrix} =0. \quad . \quad . \quad . \quad . \quad (1)$$

This is therefore the differential equation which all surfaces of revolution must satisfy. See also Salmon's Solid Geometry, § 443.

Again, if

$$ax^3+by^3+cz^3+dxyz=\text{constant}$$

is a surface of revolution, the equation (1) must be identically satisfied for all points on this surface.

Reducing, we have

$$\begin{aligned} &y^2z(nd+3lb)-z^2y(md+3lc) \\ &\quad +..+..+3ax^2(mh-ng)+..+.. \\ &\quad +dyz(mh-ng)+..+..=0 \quad . \quad . \quad . \quad . \quad . \quad (2) \end{aligned}$$

at all points on the given curve.

Therefore every coefficient in (2) vanishes. We thus find

$$f=g=h, \quad l=m=n, \quad \text{and} \quad a=b=c=-\frac{d}{3},$$

which last is the condition required. The surface of revolution of the given form is therefore

$$x^3+y^3+z^3-3xyz=\text{constant};$$

and its axis of revolution is

$$x=y=z.$$

Tuesday, November 9, 1880.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

Morning, 10 *to* 1.

DIFFERENTIAL AND INTEGRAL CALCULUS.

1. *Define Bernoulli's numbers, and expand* $\tan x$ *and* $\sec x$ *in series of powers of* x *whose coefficients are expressed by Bernoulli's numbers.*

[Boole's Finite Differences, ch. v. § 2 (5), ch. vi. § 2 (7).]

Though $\operatorname{cosec} x$ may be expanded in the manner required, $\sec x$ can only be indirectly so treated; and the result is very complex. The general form of the expansion for $\sec x$ depends on numbers analogous to Bernoulli's, and called Euler's numbers. See Boole, ch. vi. §§ 5–17.

2. *If* $z=\int_0^x (x-y)^{n-1}f(y)dx$, *find the value of* $\frac{d^n z}{dx^n}$.

We have, in the usual manner [Williamson's Integral Calculus, § 114],

$$\frac{dz}{dx}=\int_0^x \frac{d}{dx}(x-y)^{n-1}f(y)dy+[(x-y)^{n-1}f(y)];$$

in the latter expression $y=x$. This gives

$$\frac{dz}{dx}=(n-1)\int_0^x (x-y)^{n-2}f(y)dy.$$

We have thus reduced the first differential coefficient of z to a form similar to that of z.

Proceeding with the other differential coefficients, we at length obtain

$$\frac{d^{n-1}z}{dx^{n-1}}=(n-1)(n-2)\ldots.1\,.\int_0^x f(y)dy;$$

and therefore

$$\frac{d^n z}{dx^n} = (n-1)(n-2)\ldots 1 \,.\, f(x).$$

3. *If a right line move in any manner in a plane, prove that the centres of curvature of the paths described by the different points on the line in any position lie on a conic.*

We may represent the motion of any plane figure, and therefore of any right line, by supposing it fixed to one curve which rolls on another fixed curve [Williamson's Differential Calculus, § 295.]

Let O be the point of contact of these curves at any instant, P a point on the given line, then OP is the normal to P's path. Let C be the centre of curvature of P's path, CP = R, OC = ρ, OP = r, θ the angle between the normal to the two curves and OP, d the diameter of the circle of inflexions. Then [Williamson, §§ 287–290]

$$(R-r)r = Rd\cos\theta.$$

Now the polar coordinates of C, with O as origin and the normal as initial line, are ρ and θ; and $R = r-\rho$; hence

$$\rho r = (\rho - r)\, d\cos\theta. \quad . \quad . \quad . \quad . \quad . \quad (1)$$

But, since P lies in a fixed straight line, r and θ must be connected by the equation $r = c\sec(\theta-\alpha)$, where c and α are constants. Hence, substituting for r in (1) the locus of C is found to be

$$\rho c = \{\rho\cos(\theta-\alpha) - c\} d\cos\theta,$$

or
$$(x^2+y^2)c = \{x\cos\alpha + y\sin\alpha - c\} dx,$$

which represents a conic section.

4. *If in $u_1, u_2, \ldots u_n$, which are functions of n independent variables, the variables $x_1, x_2, \ldots x_n$ be* linearly *transformed, i. e. so that*

$$x_1 = a_1 X_1 + a_2 X_2 + \ldots + a_n X_n; \text{ \&c.},$$

prove that the determinant

$$\begin{vmatrix} \frac{du_1}{dx_1} & \frac{du_1}{dx_2} \cdots \frac{du_1}{dx_n} \\ \frac{du_2}{dx_1} & \frac{du_2}{dx_2} \cdots \frac{du_2}{dx_n} \\ \cdots & \cdots \\ \frac{du_n}{dx_1} & \frac{du_n}{dx_2} \cdots \frac{du_n}{dx_n} \end{vmatrix}$$

is unaltered by the transformation, provided

$$x_1^2+x_2^2+\ldots+x_n^2=X_1^2+X_2^2+\ldots+X_n^2.$$

Making the substitution, of which the following is a type,

$$\frac{du}{dX_1}=a_1\frac{du}{dx_1}+b_1\frac{du}{dx_2}+c_1\frac{du}{dx_3}+\ldots$$

we find that the transformed determinant is equal to the original determinant multiplied by the modulus of the coefficients of transformation [Salmon's Higher Algebra, § 43], and this modulus is equal to unity, which proves the required result.

5. *If through each point on a curve a right line be drawn of given length, making a constant angle with the normal at the point; prove that the normal to the locus of the extremities of the lines so drawn passes through the centre of curvature of the point on the given curve.*

This result may be easily established by means of the differential calculus applied to Cartesian coordinates, or it may be geometrically proved thus:—Let PO, QO be normals to the given curve at consecutive points P, Q; then O is ultimately the centre of curvature at P, and PO=QO. Also let PT, QU be the lines of constant length. Join TO, UO, TU. Then the triangles PTO, QUO are ultimately equal in all respects from the conditions of the question, and TO=UO. Thus ultimately TO and UO are both at right angles to TU, or are normals to the locus of their extremities.

NOTE. The equivalence of TO and UO is as far as the second order of small quantities.

6. *Find the values of the definite integrals*

$$\int_0^1 \frac{(x^a - x^b)\,dx}{\log x} \quad \text{and} \quad \int_0^\infty \frac{x^{2m}\,dx}{x^{2n}+1},$$

where n and m are both positive integers and $n > m$.

In the first integral, by writing $\log x = y$, we obtain a case of Frullani's Theorem, for which see Williamson's Integral Calculus, § 119. We may also integrate by a similar method to that used in question 5, page 169.

Or we may proceed thus:—Let the required integral be called u; then, differentiating it with regard to a,

$$\frac{du}{da} = \int_0^1 x^a\,dx = \frac{1}{a+1}, \text{ by integrating. . . . (1)}$$

Similarly,

$$\frac{du}{db} = -\frac{1}{b+1}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Hence, from (1) and (2), by integration,

$$u = \log(a+1) - \log(b+1) + C.$$

To determine C, we observe that, when $a=b$, $u=0$; thus

$$u = \log(a+1) - \log(b+1).$$

The second integral in the question is worked out in Williamson's Integral Calculus, § 108.

7. *In a curve, if* $s^5 = px^4$, *where s denotes its length measured from the origin, find the equation of the curve in terms of x and y.*

We have immediately

$$\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2 = \frac{16}{25}\left(\frac{p}{x}\right)^{\frac{2}{5}}.$$

Now let

$$5^5 y = \eta^5, \quad 5^5 x = \xi^5, \quad 4^5 p = q^5; \quad . \quad . \quad . \quad . \quad (1)$$

we get $$1+\frac{\eta^6}{\xi^6}\left(\frac{d\eta}{d\xi}\right)^2=\left(\frac{q}{\xi}\right)^2,$$

or $$\eta^4\frac{d\eta}{d\xi}=\xi^2\sqrt{q^2-\xi^2}=\xi\{q^2-(q^2-\xi^2)\}\sqrt{q^2-\xi^2};$$

$$\therefore\quad \frac{\eta^5}{5}=\tfrac{1}{5}(q^2-\xi^2)^{\frac{1}{2}}-\tfrac{1}{3}q^2(q^2-\xi^2)^{\frac{3}{2}}+\text{const.}$$

And since the curve passes through the origin, we have

$$3\eta^5=3(q^2-\xi^2)^{\frac{1}{2}}-5q^2(q^2-\xi^2)^{\frac{3}{2}}+2q^5.$$

This may be transformed back to an equation between x, y, and p by means of equation (1).

8. *In the definite integral* $\int_{x_0}^{X}\frac{(x+c)dx}{x^2+a^2}$, *if* X *approaches to* $+\infty$ *and* x_0 *to* $-\infty$, *show that the value of the integral becomes indeterminate.*

For the same limits, show that $\int_{x_0}^{X}\frac{(x^2+c^2)dx}{(x^2+a^2)(x^2+b^2)}$ *is determinate, and find its value.*

State the general principles of which the preceding results are particular cases.

We have

$$\int_{x_0}^{X}\frac{(x+c)dx}{x^2+a^2}=\left[\tfrac{1}{2}\log(x^2+a^2)+c\tan^{-1}\frac{x}{a}\right]_{x_0}^{X}$$

$$=\tfrac{1}{2}\log\frac{X^2+a^2}{x_0^2+a^2}+\frac{c}{a}\left\{\tan^{-1}\frac{X}{a}-\tan^{-1}\frac{x_0}{a}\right\}.$$

Now when $X=\infty$, $x_0=-\infty$, the second member of the preceding result is $c\pi\div a$; but the first is perfectly indeterminate because we do not know the limiting value of the ratio $X:x_0$.

Whatever this limit may be, however, the whole result can be always written

$$\tfrac{1}{2}\log\frac{X^2}{x_0^2}+\frac{c\pi}{a}.$$

Again, the second integral is

$$\int_{x_0}^{X}\left\{\frac{a^2-c^2}{x^2+a^2}-\frac{b^2-c^2}{x^2+b^2}\right\}\frac{dx}{a^2-b^2}$$

$$=\left[\frac{(a^2-c^2)}{a}\tan^{-1}\frac{x}{a}-\frac{(b^2-c^2)}{b}\tan^{-1}\frac{x}{b}\right]_{x_0}^{X}\div(a^2-b^2)$$

$$=\frac{\pi(ab+c^2)}{ab(a+b)}.$$

For fuller details see Williamson's Integral Calculus, §§ 100–107.

9. *Solve the differential equations*

(*a*) $y-x\dfrac{dy}{dx}=nx\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$,

(*b*) $x^4\dfrac{d^2y}{dx^2}+a^2y=0$,

(*c*) $\dfrac{d^2z}{dx^2}\left(\dfrac{dz}{dy}\right)^2-2\dfrac{d^2z}{dx\,dy}\dfrac{dz}{dx}\dfrac{dz}{dy}+\dfrac{d^2z}{dy^2}\left(\dfrac{dz}{dx}\right)^2=0.$

In the first equation let $y=xv$, then we find, on separating the variables,

$$\frac{\sqrt{1-n^2+v^2}-nv}{1+v^2}\cdot\frac{dv}{dx}=\frac{n^2}{x},$$

or $$\frac{dv}{\sqrt{1+v^2-n^2}}-\frac{n^2dv}{(1+v^2)\sqrt{1+v^2-n^2}}-\frac{nvdv}{1+v^2}=\frac{n^2dx}{x};$$

$$\therefore\quad \log\{v+\sqrt{1+v^2-n^2}\}-n\log\{n\sin\theta+\sqrt{1-n^2\cos^2\theta}\}$$
$$-\tfrac{1}{2}n\log(1+v^2)=n^2\log x+C,$$

where $\tan\theta = v$; this reduces to

$$\log\{v+\sqrt{1+v^2-n^2}\}-n\log\{nv+\sqrt{1+v^2-n^2}\} = n^2\log x+C.$$

In equation (*b*) let $\xi x = a$, and transform to ξ as independent variable; the equation becomes

$$\xi\frac{d^2y}{d\xi^2}+2\frac{dy}{d\xi}+y\xi=0,$$

or

$$\frac{d^2}{d\xi^2}(y\xi)+y\xi \quad =0,$$

a linear equation of which the solution is known to be

$$y\xi = A\sin(\xi+B).$$

The solution of the original equation is therefore

$$y = Ax\sin\left(\frac{a}{x}+B\right).$$

It may also be solved by writing

$$z\,dx = d(\log y),$$

in which case it becomes an equation of Riccati: see Boole's Differential Equations, ch. vi. § 1, ch. x. ex. (27).

Equation (*c*) may be integrated by the ordinary method [Boole, ch. xv. §§ 2–4], or thus:—We have

$$rq^2-2spq+tp^2=0,$$

or

$$q(rq-sp)=p(sq-tp);$$

hence

$$\frac{dz}{dx}\cdot\frac{d}{dy}\left(\frac{p}{q}\right)=\frac{dz}{dy}\cdot\frac{d}{dx}\left(\frac{p}{q}\right).$$

But this is the condition that the quantities $\frac{p}{q}$ and z are functions of each other [Boole, ch. ii. § 1].

Therefore

$$p+qf(z)=0;$$

and on integrating this by the ordinary method of Lagrange, we have

$$y = xf(z) + \psi(z),$$

which is the integral required.

We may also solve easily by changing the variables to (x, z), by which means we obtain an equation of Clairaut, whose solution is $y = cx + \phi(c)$, where c is an arbitrary function of z, a result agreeing with that already obtained.

10. *Find the value of the multiple integral*

$$\iiint \sqrt{\frac{1-(x^2+y^2+z^2)}{1+x^2+y^2+z^2}}\, dx\, dy\, dz,$$

extended to all values of x, y, z, subject to the condition that

$$x^2+y^2+z^2 > 1.$$

The subject of integration depending only on the distance of (x, y, z) from the origin, and the integral being taken throughout a sphere of radius unity, with its centre at the origin, we may at once write the integral

$$4\pi \int_0 \sqrt{\frac{1-r^2}{1+r^2}} \cdot r^2\, dr,$$

where the element of volume is now a thin spherical shell of radii r, $r+dr$.

Now let $r = \sin\theta$, and call the integral u; we have

$$u = 4\pi \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta \cos^2\theta}{\sqrt{1+\sin^2\theta}} \cdot d\theta$$

$$= 4\pi \left[\sin\theta \cos\theta \sqrt{1+\sin^2\theta}\right]_0^{\frac{\pi}{2}} d\theta - 4\pi \int \cos 2\theta \sqrt{1+\sin^2\theta}\, d\theta$$

$$= 4\pi \int (2\sin^2\theta - 1)\sqrt{1+\sin^2\theta} \cdot d\theta$$

$$= 4\pi \int \frac{(1-2\cos^2\theta)(1+\sin^2\theta)d\theta}{\sqrt{1+\sin^2\theta}}$$

$$= 4\pi \int \frac{1-2\cos^2\theta+\sin^2\theta}{\sqrt{1+\sin^2\theta}} d\theta - 2u;$$

reducing we have

$$3u = 4\pi \int 3\sqrt{1+\sin^2\theta}\,.\,d\theta - 4\pi \int \frac{4\,d\theta}{\sqrt{1+\sin^2\theta}}.$$

Now change θ into $\frac{1}{2}\pi - \theta$, and we at once obtain

$$3u = 12\pi\sqrt{2}\int_0^{\frac{\pi}{2}} \sqrt{1-\tfrac{1}{2}\sin^2\theta}\,.\,d\theta - 8\pi\sqrt{2}\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$$

$$= 4\pi\sqrt{2}\{3\mathrm{E}_1(\sqrt{\tfrac{1}{2}}) - 2\mathrm{F}_1(\sqrt{\tfrac{1}{2}})\}.$$

See Cayley's Elliptic Functions, § 10.

11. *In the equation*

$$\frac{d^n y}{dx^n} + \mathrm{P}_1 \frac{d^{n-1}y}{dx^{n-1}} + \mathrm{P}_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots + \mathrm{P}_{n-1}\frac{dy}{dx} + \mathrm{P}_n y = \mathrm{X},$$

if $\mathrm{P}_1, \mathrm{P}_2, \ldots \mathrm{P}_n, \mathrm{X}$ *be functions of* x *solely; prove that the integral is of the form*

$$y = u_1 \int u_2\,dx \int u_3\,dx \ldots \int \frac{\mathrm{X}\,dx}{u_1 u_2 \ldots u_n};$$

and show how $u_1, u_2, \ldots u_n$ *are to be determined.*

Let $y = u_1 z_1$; then, on substituting in the given differential equation, we get

$$u_1 \frac{d^n z_1}{dx^n} + \mathrm{Q}_1 \frac{d^{n-1}z_1}{dx^{n-1}} + \ldots + \mathrm{Q}_{n-1}\frac{dz_1}{dx}$$

$$+ z_1 \left\{ \frac{d^n u_1}{dx^n} + \mathrm{P}_1 \frac{d^{n-1}u_1}{dx_{n-1}} + \ldots + \mathrm{P}_{n-1}\frac{du_1}{dx} + \mathrm{P}_n u_1 \right\} = \mathrm{X}.$$

Now if the expression in brackets vanishes, which it will if u_1 satisfies the differential equation made by equating this expression to zero, we have left a linear equation of the $(n-1)$th degree in $\frac{dz_1}{dx}$; writing then $\frac{dz_1}{dx}=y_1$, and $Q_r=u_1R_r$, we have

$$\frac{d^{n-1}y_1}{dx^{n-1}}+R_1\frac{d^{n-2}y_1}{dx^{n-2}}+\ldots+R_{n-1}y_1=\frac{X}{u_1}.$$

But this equation is of the same form as the original equation, and one degree lower; proceeding then in the same way, we at last get

$$y_n=\frac{X}{u_1u_2\ldots\ldots u_n}.$$

But
$$y=u_1z_1=u_1\int y_1dx,$$
$$y_1=u_2z_2=u_2\int y_2dx,$$
$$\ldots\ldots\ldots$$
$$y_{n-1}=u_nz_n=u_n\int y_ndx.$$

Whence
$$y=u_1\int u_2dx\int u_3dx\ldots\int\frac{X\,dx}{u_1u_2\ldots u_n}.$$

Monday, November 7, 1881.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
BENJAMIN WILLIAMSON, Esq., M.A., F.R.S.

Morning, 10 *to* 1.

ALGEBRA AND TRIGONOMETRY.

1. *Prove, for all real values of the constants, that the roots of the equation*

$$\begin{vmatrix} x-a & h & g \\ h & x-b & f \\ g & f & x-c \end{vmatrix} = 0$$

are always real.

[Salmon's Solid Geometry, § 83; Higher Algebra, §§ 46–53.]

2. *If* $\alpha, \beta, \gamma, \delta$ *be the roots of the equation*

$$x^4+px^3+qx^2+rx+s=0,$$

find, in terms of the coefficients, the value of

$$\Sigma(\alpha-\beta)^2(\gamma-\delta)^2.$$

The expression required may be found by means of the differential equation which symmetrical functions of the differences of the roots satisfy. Or we may proceed as follows:—

$\Sigma(\alpha-\beta)^2(\gamma-\delta)^2$ consists of three separate kinds of terms, and when multiplied out and arranged is easily seen to be

$$2\Sigma\alpha^2\beta^2-2\Sigma\alpha\beta\gamma^2+12\Sigma\alpha\beta\gamma\delta.$$

Now

$$\Sigma\alpha^2\beta^2=(\Sigma\alpha\beta)^2-2\Sigma\alpha\,.\,\Sigma\alpha\beta\gamma+2\Sigma\alpha\beta\gamma\delta,$$

$$\Sigma\alpha\beta\gamma^2=\Sigma\alpha\,.\,\Sigma\alpha\beta\gamma-4\Sigma\alpha\beta\gamma\delta.$$

Thus the given function is

$$2(\Sigma a\beta)^2 - 6\Sigma a \,.\, \Sigma a\beta\gamma + 24\Sigma a\beta\gamma\delta,$$

or

$$2q^2 - 6rp + 24s.$$

3. *State what is meant by the Jacobian of a system of equations, and show that if the equations are satisfied by a common system of values of the variables, that system will satisfy the Jacobian also.*

[Salmon's Higher Algebra, §§ 88, 89.]

4. *Find the simplest form of the expression*

$$\frac{\alpha^3}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\alpha-\epsilon)} + \frac{\beta^3}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)(\beta-\epsilon)} + \ldots\ldots + \frac{\epsilon^3}{(\epsilon-\alpha)(\epsilon-\beta)(\epsilon-\gamma)(\epsilon-\delta)}.$$

Let $f(x) = (x-\alpha)(x-\beta)\ldots\ldots(x-\epsilon)$,

and let $\phi(x)$ be an expression of degree not above the fourth; then we may split up $\phi(x) \div f(x)$ into partial fractions. We have, as usual,

$$\frac{\phi(x)}{f(x)} = \frac{\phi(\alpha)}{(x-\alpha)(\alpha-\beta)(\alpha-\gamma)\ldots\ldots(\alpha-\epsilon)} + \text{similar terms.}$$

Now put $x=0$, and we have

$$\frac{\phi(0)}{f(0)} = \frac{\phi(\alpha)}{-\alpha(\alpha-\beta)(\alpha-\gamma)\ldots\ldots(\alpha-\epsilon)} + \frac{\phi(\beta)}{-\beta(\beta-\alpha)(\beta-\gamma)\ldots\ldots(\beta-\epsilon)} + \&c.$$

For the given example we take $\phi(x) = x^4$ and $\phi(0) = 0$, whence

$$\frac{\alpha^3}{(\alpha-\beta)(\alpha-\gamma)\ldots(\alpha-\epsilon)} + \frac{\beta^3}{(\beta-\alpha)(\beta-\gamma)\ldots(\beta-\epsilon)} + \&c. = 0.$$

The more general theorem, which can be proved in the same way, is found by resolving $x\phi(x) \div f(x)$ into partial fractions, where $f(x)$ is of the nth degree and $\phi(x)$ not higher than the $(n-2)$th. In this case

$$\frac{\phi(\alpha)}{(\alpha-\beta)(\alpha-\gamma)\ldots.} + \frac{\phi(\beta)}{(\beta-\alpha)(\beta-\gamma)\ldots.} + \&c. = 0.$$

5. *If* Δ *be the discriminant of the quantic*

$$(a_0, a_1, a_2, \ldots . a_{n-1}, a_n)(x, y)^n,$$

prove that we have

$$a_1\frac{d\Delta}{da_0} + 2a_2\frac{d\Delta}{da_1} + 3a_3\frac{d\Delta}{da^2} + \ldots . + na_n\frac{d\Delta}{da_{n-1}} = 0;$$

and apply to find the discriminant of the cubic

$$a_0x^3 + a_1x^2y + a_2xy^2 + a_3y^3.$$

[Salmon's Higher Algebra, §§ 63, 112.]

6. *If the sum of the arcs joining a point to two fixed points on a sphere is constant, prove that the locus of the point is a sphero-conic; and state the corresponding theorem obtained by polar reciprocation.*

A solution of this question is given in Salmon's Solid Geometry, § 249. We may also proceed thus :—Let S, H be the fixed points, C the middle point of SH, P any point on the locus, PN drawn at right angles to SH.

Suppose that a tangent plane is drawn at C, and that s, h, p, n are the points in the plane where the radii OS, OH. . . . meet it. Then the locus of p is a plane section of the cone on which the locus of P lies; and if we prove that p lies on a conic, then P will lie on a cone of the second degree, and therefore on a sphero-conic.

Now let

$$PN = y, \quad CN = x, \quad CS = c, \quad SP + HP = r + r' = 2a,$$

$$pn = \eta, \quad Cn = \xi,$$

and the radius of the sphere = unity.

Then it is easy to see that the following relations hold between ξ, η, x, y:

$$\xi = \tan x, \quad \eta = \sec x \tan y. \quad . \quad . \quad . \quad . \quad (1)$$

Now $\cos r = \cos y \cos (c - x)$,

$\cos r' = \cos y \cos (c + x)$;

$\therefore \quad \cos r + \cos r' = 2 \cos \frac{1}{2}(r - r') \cos a = 2 \cos y \cos c \cos x,$

and $\cos r - \cos r' = 2 \sin \frac{1}{2}(r' - r) \sin a = 2 \cos y \sin c \sin x.$

Hence, multiplying these equations respectively by $\sin a$ and $\cos a$, squaring and adding, we obtain

$$\cos^2 a \sin^2 a = \cos^2 y (\sin^2 a \cos^2 c \cos^2 x + \cos^2 a \sin^2 c \sin^2 x). \quad (2)$$

Again, from (1) we have

$$1 + \xi^2 + \eta^2 = \sec^2 x + \sec^2 x \tan^2 y = \sec^2 x \sec^2 y. \quad (3)$$

Hence multiplying (2) and (3) together, we have

$$\cos^2 a \sin^2 a (1 + \xi^2 + \eta^2) = \sin^2 a \cos^2 c + \cos^2 a \sin^2 c \tan^2 x,$$

or $\quad \sin^2 a \, \eta^2 + (\sin^2 a - \sin^2 c) \xi^2 = \tan^2 a \cos^2 c - \sin^2 a.$

Thus the locus of p is a conic, and therefore that of P is a sphero-conic.

7. *O and O′ are two diametrically opposite points on the surface of a sphere. If the point* P *move round the circumference of a lesser circle of the sphere, prove that the line* OP *will trace out a circle on the tangent plane at the point* O′.

Let the plane of the paper pass through O, O′, and the diameter AB of the small circle; join OA, OB, and produce them to meet the tangent plane at O′ in C, D. Join O′B. Then both the plane of the small circle and the tangent plane are perpendicular to the paper; also since $\angle$ O′BO is a right angle, we have

$$\angle O'DO = \angle OO'B = \angle OAB.$$

Therefore the tangent plane makes with the cone a sub-contrary section to that of the plane of the small circle; and therefore the section by the tangent plane is also a circle. See Besant's Conic Sections, § 234.

8. *If* L, M *and* L', M' *be the points of intersection with a lesser circle of any two great circles drawn from a fixed point on the surface of a sphere, find the locus of the point of intersection of the great circles* LL', MM'.

Suppose the whole figure to be projected with the centre of the sphere as vertex, and the tangent plane at O, the centre of the small circle, as plane of projection. Then the small circle projects into a circle on the tangent plane; and the points L, M, L', M', and the fixed point P project into points l, m, l', m', p.

Now suppose that lm', $l'm$ meet in x, and ll', mm' in y, then xy, by harmonic properties of a quadrilateral, divides pll', pmm' harmonically; hence xy must be the polar of p, and thus the locus of x is the chord of contact of tangents from p; and therefore the locus of the corresponding point X on the sphere is a great circle which meets the small circle at the points of contact of the great circle tangents to the small circle from P.

9. *If* α, β, γ *be the arcs joining the middle points of the sides of a spherical triangle respectively, prove that*

$$\frac{\cos\alpha}{\cos\frac{1}{2}a}=\frac{\cos\beta}{\cos\frac{1}{2}b}=\frac{\cos\gamma}{\cos\frac{1}{2}c}.$$

Let ABC be the triangle, P, Q_1 R the middle points of a, b, c respectively, then $QR=\alpha$, and

$$\cos\alpha=\sin\frac{b}{2}\sin\frac{c}{2}\cos A+\cos\frac{b}{2}\cos\frac{c}{2};$$

but $$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c}.$$

Hence $$\cos\alpha = \frac{\cos a - \cos b \cos c}{4\cos\frac{b}{2}\cos\frac{c}{2}} + \cos\frac{b}{2}\cos\frac{c}{2}$$

$$= \frac{\cos a + 2\cos^2\frac{b}{2} + 2\cos^2\frac{c}{2} - 1}{4\cos\frac{b}{2}\cos\frac{c}{2}}$$

$$= \frac{\cos\frac{a}{2} + \cos^2\frac{b}{2} + \cos^2\frac{c}{2} - 1}{2\cos\frac{b}{2}\cos\frac{c}{2}};$$

and similar expressions may be found for $\cos\beta$ and $\cos\gamma$, thus

$$\frac{\cos\alpha}{\cos\frac{a}{2}} = \frac{\cos\beta}{\cos\frac{b}{2}} = \frac{\cos\gamma}{\cos\frac{c}{2}}.$$

10. *If* A, B, C, D *be four points on a lesser circle, prove the relation*

$$\tan\tfrac{1}{2}\mathrm{AD}\sin\mathrm{BDC} + \tan\tfrac{1}{2}\mathrm{BD}\sin\mathrm{CDA} + \tan\tfrac{1}{2}\mathrm{CD}\sin\mathrm{ADB} = 0,$$

where the angles are considered with their proper signs, i. e. $\mathrm{ADB} = -\mathrm{BDA}$ *&c.*

Let O be the centre of the circle, and let the radius OD be ρ; also call the angles which DA, DB, DC make with DO, measured on the same side, α, β, γ respectively; from O draw arcs OP, OQ, OR at right angles to AD, BD, CD, which they respectively bisect.

Then $\angle\mathrm{BDC} = \gamma - \beta$, $\angle\mathrm{CDA} = \alpha - \gamma$, $\angle\mathrm{ADB} = \beta - \alpha$.

Also, from the right-angled triangle POD,

$$\tan\mathrm{PD} = \tan\rho\cos\alpha,$$

or

$$\tan\tfrac{1}{2}\mathrm{AD} = \tan\rho\cos\alpha;$$

similarly for the other sides.

Thus the given expression becomes

$$\tan\rho\{\cos\alpha\sin(\gamma-\beta)+\cos\beta\sin(\alpha-\gamma)+\cos\gamma\sin(\beta-\alpha)\},$$

which vanishes.

Afternoon, 3 to 6.

PURE AND COORDINATE GEOMETRY.

1. *Being given five points on an ellipse, show how to determine, by a geometrical construction, the position of its axes major and minor, and also their lengths.*

Let A, B, C, D, E be the given points, and suppose AB, DE to meet in X; through X draw XY parallel to BC, meeting CD produced in Y. Join AY, meeting EZ parallel to BC in F; then F is on the conic; for the intersections of AB, DE; BC, EF; CD, FA are on the straight line XY. Bisect the parallel chords BC, EF, and join the points obtained; we thus have a diameter. Similarly we can get another diameter, and thus we have the centre; and we may then proceed as in Besant's Conic Sections, § 219.

2. *Define the Hessian* (H) *of any curve, and prove that the curve represented by the equation* $H=0$ *passes through the points of inflexion of the curve, and also through its double points.*

[Salmon's Higher Plane Curves, §§ 70, 71, 74.]

3. *If any conic be described through four fixed points on a curve of the third degree, prove that the chord joining the two remaining points of intersection of the curves passes through a fixed point.*

Let the equations to two pairs of lines through the four fixed points be $\alpha\beta=0$, and $\gamma\delta=0$; then since the cubic passes through the intersections of $\alpha\beta=0$, $\gamma\delta=0$, its equation must be $\alpha\beta u+\gamma\delta v=0$, where u and v are linear functions of x and y. Now let any conic through the four

points be $l\alpha\beta+m\gamma\delta=0$, then its six intersections with the given cubic are given by

$$\alpha\beta u+\gamma\delta v=0,$$
$$l\alpha\beta+m\gamma\delta=0.$$

Therefore either $\alpha\beta$ and $\gamma\delta=0$, giving the four fixed points, or $mu-lv=0$. This last equation gives the line on which the two remaining points lie, and which passes through the fixed point $u=0$, $v=0$.

4. *What general theorems can be deduced from the following?*

1. *The tangent to a circle is at right angles to the radius drawn to the point of contact.*

2. *The locus of the intersection of two tangents to a central conic, which intersect at right angles, is a circle.*

(*a*) *By the method of reciprocal polars*, (*b*) *by the method of projection.*

Taking the theorem (1) and reciprocating, with regard to a circle, we find that

"The portion of the tangent to any conic section intercepted between the point of contact and the directrix, subtends a right angle at the focus."

Taking (2) and reciprocating, with regard to a concentric circle, we see that

"The chords joining the extremities of two orthogonal diameters in a conic envelopes a concentric circle."

Or, reciprocating with regard to any point,

"A chord of a conic subtending a right angle at a fixed point envelopes a conic of which that point is a focus."

For the results of projection see Salmon's Conic Sections, § 356, exx. 1 and 5.

5. *Find the locus of the origin of reciprocation for which the reciprocal polar of a surface of the second degree is a surface of revolution.*

Salmon's Solid Geometry, § 155; or thus:—

Let $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1$ be the given surface; then we require the locus of a point P(f, g, h) such that the polar reciprocal of the enveloping cone from (f, g, h) may be a right cone.

Any tangent plane to the quadric may be written

$$lx+my+nz=\sqrt{l^2a+m^2b+n^2c};$$

and if this passes through (f, g, h), then

$$lf+mg+nh=\sqrt{l^2a+m^2b+n^2c}.$$

Thus the reciprocal cone having l, m, n for the direction cosines of its generating lines, must have for equation, referred to P as origin,

$$(fx+gy+hz)^2=x^2a+y^2b+z^2c,$$

or $$x^2(f^2-a)+y^2(g^2-b)+z^2(h^2-c)+2yzgh+2zxhf+2xyfg=0.$$

Now this will be found to be a cone of revolution only when (f, g, h) is on one of the focal conics by applying the usual test.

6. *If a curve of the third degree has a double point, prove that its equation may be written in the form*

$$(x^2\pm y^2)z=x^3.$$

Explain the relations of the lines x, y, z with the curve, and find the number of its real points of inflexion according as the double point is a node or a conjugate point.

[Salmon's Higher Plane Curves, § 215.]

7. *If a cone be circumscribed to a surface of the second degree, prove that its axes are the normals to the system of confocal surfaces passing through its vertex; and that its focal lines are the generatrices of the confocal hyperboloid of one sheet.*

[Salmon's Solid Geometry, §§ 167, 169, 173, 175.]

8. *Find the functional and partial differential equations of gauche, developable, and tubular surfaces respectively.*

[Salmon's Solid Geometry, ch. xiii.]

9. *Find the general equation of geodesic lines on a surface* $F(x,y,z)=0$*; and prove that at each point of a geodesic on an ellipsoid* pD *is constant, where* p *is the central perpendicular on the tangent plane at the point, and* D *is the diameter of the surface drawn parallel to the tangent to the geodesic at the same point.*

The principal normal of a geodesic coincides with the normal of the surface at the point considered; the differential equation of geodesic lines is therefore

$$\frac{\frac{d^2x}{ds^2}}{\frac{dF}{dx}}=\frac{\frac{d^2y}{ds^2}}{\frac{dF}{dy}}=\frac{\frac{d^2z}{ds^2}}{\frac{dF}{dz}}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Now consider the ellipsoid whose equation is

$$\alpha x^2+\beta y^2+\gamma z^2-1=F=0.$$

Then, denoting differential coefficients with regard to s, a geodesic arc, by accents we have

$$p^2(\alpha^2x^2+\beta^2y^2+\gamma^2z^2)=D^2(\alpha x'^2+\beta y'^2+\gamma z'^2)=1. \quad . \quad (2)$$

Differentiating (1) twice and (2) once,

$$\alpha x'^2+\beta y'^2+\gamma z'^2+\alpha xx''+\beta yy''+\gamma zz''=0, \quad . \quad . \quad . \quad (3)$$

$$\alpha^2xx'+\beta^2yy'+\gamma^2zz'=-p^{-3}p', \quad . \quad (4)$$

$$\alpha x'x''+\beta y'y''+\gamma z'z''=-D^{-3}D'. \quad . \quad (5)$$

But from (1)

$$\frac{x''}{\alpha x}=\frac{y''}{\beta y}=\frac{z''}{\gamma z}=\frac{\alpha x'x''+\beta y'y''+\gamma z'z''}{\alpha^2xx'+\beta^2yy'+\gamma^2zz'}.$$

$$=\frac{\alpha xx''+\beta yy''+\gamma zz''}{\alpha^2x^2+\beta^2y^2+\gamma^2z^2}.$$

Now, applying (2), (3), (4), (5) to the last equations,

$$\frac{D^{-3}D'}{p^{-3}p'} = -\frac{p^2}{D^2}, \quad \text{or} \quad pD' + Dp' = 0;$$

whence $pD = \text{constant}$.

10. *Prove that the anharmonic ratio of four tangent planes passing through a generator of a ruled surface is equal to that of their points of contact.*

[Salmon's Solid Geometry, § 470.]

Tuesday, November 8, 1881.

Examiners, { Dr. JOHN HOPKINSON, M.A., F.R.S.
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Morning, 10 *to* 1.

DIFFERENTIAL AND INTEGRAL CALCULUS.

1. U *is a function of* ξ, η, ζ. *If we are given*

$$x\xi = y\eta = z\zeta = 1 + u,$$

where u *is a homogeneous function of the degree* n *in* x, y, z, *prove that*

$$x\frac{dU}{dx} + y\frac{dU}{dy} + z\frac{dU}{dz} + \frac{n}{x}\frac{dU}{d\xi} + \frac{n}{y}\frac{dU}{d\eta} + \frac{n}{z}\frac{dU}{d\zeta}$$
$$= (n-1)\left(\xi\frac{dU}{d\xi} + \eta\frac{dU}{d\eta} + \zeta\frac{dU}{d\zeta}\right).$$

Since $x\xi = y\eta = z\zeta = 1 + u$, we have

$$\log x + \log \xi = \log y + \log \eta = \log z + \log \zeta = \log(1+u).$$

Thus

$$\left.\begin{aligned} \frac{x}{\xi}\frac{d\xi}{dx} &= \frac{x}{1+u}\frac{du}{dx}-1, \\ \frac{y}{\xi}\frac{d\xi}{dy} &= \frac{y}{1+u}\frac{du}{dy}, \\ \frac{z}{\xi}\frac{d\xi}{dz} &= \frac{z}{1+u}\frac{du}{dz}; \end{aligned}\right\} \quad \ldots\ldots \quad (1)$$

and similarly for η, ζ.

Now

$$\begin{aligned} & x\frac{dU}{dx}+y\frac{dU}{dy}+z\frac{dU}{dz}+\frac{n}{x}\frac{dU}{d\xi}+\frac{n}{y}\frac{dU}{d\eta}+\frac{n}{z}\frac{dU}{d\zeta} \\ &= \frac{dU}{d\xi}\left(x\frac{d\xi}{dx}+y\frac{d\xi}{dy}+z\frac{d\xi}{dz}\right)+..+.. \\ &\quad +\frac{n\xi}{1+u}\frac{dU}{d\xi}+\frac{n\eta}{1+u}\frac{dU}{d\eta}+\frac{n\zeta}{1+u}\frac{dU}{d\zeta} \\ &= \xi\frac{dU}{d\xi}\left(\frac{x}{1+u}\frac{du}{dx}+\frac{y}{1+u}\frac{du}{dy}+\frac{z}{1+u}\frac{du}{dz}-1+\frac{n}{1+u}\right) \\ &\quad +..\qquad +..\qquad \text{by (1).} \end{aligned}$$

Now by the theory of homogeneous functions this becomes

$$\begin{aligned} & \xi\frac{dU}{d\xi}\left(\frac{nu}{1+u}+\frac{n}{1+u}-1\right)+..+.. \\ &\quad =(n-1)\left(\xi\frac{dU}{d\xi}+\eta\frac{dU}{d\eta}+\zeta\frac{dU}{d\zeta}\right). \end{aligned}$$

2. *If all the parts of a spherical triangle vary simultaneously, prove that*

$$\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}=\tan A\,\tan B\,\tan C\,dk,$$

where $k=\dfrac{\sin a}{\sin A}$.

The total change in k made by supposing all the sides to vary simultaneously is the sum of the changes when each varies by itself; for

$$dk=\frac{dk}{da}\,.\,da+\frac{dk}{db}\,.\,db+\frac{dk}{dc}\,.\,dc.$$

Supposing, then, one of the sides (e. g. a) to vary at a time, let us find the corresponding change in k.

We have

$$\log k=\log\sin a-\log\sin A,$$

$$\cos A\sin b\sin c=\cos a-\cos b\cos c.$$

Hence $$dk=k\cot a\,da-k\cot A\,dA, \quad . \ . \ . \quad (1)$$

and $$\sin A\sin b\sin c\,dA=\sin a\,da,$$

or $$\sin B\sin c\,dA=da; \quad . \ . \ . \ . \ . \ . \ . \quad (2)$$

Therefore, by applying (1) and (2),

$$\begin{aligned}\sin A\sin B\sin C\,dk&=\cos a\sin B\sin C\,da\\&\quad-k\cos A\sin B\sin C\,dA\\&=da(\cos a\sin B\sin C-\cos A)\\&=da\cos B\cos C.\end{aligned}$$

Now, supposing all the parts to vary, we have

$$\begin{aligned}dk\sin A\sin B\sin C&=da\cos B\cos C+db\cos C\cos A\\&\quad+dc\cos A\cos B,\end{aligned}$$

or $$dk\tan A\tan B\tan C=\frac{da}{\cos A}+\frac{db}{\cos B}+\frac{dc}{\cos C}.$$

3. *If* $$u=\frac{yz}{x},\quad v=\frac{zx}{y},\quad w=\frac{xy}{z},$$

find the value of the determinant

$$\begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} & \frac{du}{dz} \\ \frac{dv}{dx} & \frac{dv}{dy} & \frac{dv}{dz} \\ \frac{dw}{dx} & \frac{dw}{dy} & \frac{dw}{dz} \end{vmatrix}.$$

We have

$$\frac{du}{dx} = -\frac{u}{x}, \quad \frac{du}{dy} = \frac{u}{y}, \quad \frac{du}{dz} = \frac{u}{z}, \text{ and so on;}$$

the determinant therefore becomes

$$\begin{vmatrix} -\frac{u}{x} & \frac{u}{y} & \frac{u}{z} \\ \frac{v}{x} & -\frac{v}{y} & \frac{v}{z} \\ \frac{w}{x} & \frac{w}{y} & -\frac{w}{z} \end{vmatrix} \quad \text{or} \quad \frac{uvw}{xyz} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}.$$

This at once reduces to $\frac{4uvw}{xyz}$ or $4xyz$.

4. *If z be a function of x and y, prove that the equation*

$$\frac{d}{dy}\left(f(z)\frac{dz}{dx}\right) = \frac{d}{dx}\left(f(z)\frac{dz}{dy}\right)$$

still holds if z be of the form $u+iv$, where i stands for the maginary unit $\sqrt{-1}$.

If $z=u+iv$, $f(z)$ will be of the form $U+iV$, and the left-hand member becomes

$$\frac{d}{dy}\left\{\left(U\frac{du}{dx} - V\frac{dv}{dx}\right) + i\left(V\frac{du}{dx} + U\frac{dv}{dx}\right)\right\}$$

$$= \frac{dU}{dy}\frac{du}{dx} - \frac{dV}{dy}\frac{dv}{dx} + U\frac{d^2u}{dx\,dy} - V\frac{d^2v}{dx\,dy}$$

$$+ i\left(\frac{dV}{dy}\frac{du}{dx} + \frac{dU}{dy}\frac{dv}{dx} + V\frac{d^2u}{dx\,dy} + U\frac{d^2v}{dx\,dy}\right). \quad . \quad . \quad (1)$$

Now $d(U+iV) = d \cdot f(z) = f'(z)(du+idv)$;

$$\therefore \quad \frac{\frac{dU}{dx} + i \cdot \frac{dV}{dx}}{\frac{dU}{dy} + i \cdot \frac{dV}{dy}} = \frac{\frac{du}{dx} + i \cdot \frac{dv}{dx}}{\frac{du}{dy} + i \cdot \frac{dv}{dy}};$$

and, equating real and imaginary parts, we find that the expression (1) reduces to the corresponding expression for the right-hand side of the equality, which is thus proved.

5. *A is a fixed point in the circumference of a circle. The points* P, Q *are taken so that arc* AP $= m$ *arc* AQ, *where* m *is a constant. Prove that the envelope of* PQ *is an epicycloid or a hypocycloid according as the arcs* AP, AQ *are measured in the same or opposite directions from the point* A.

Let the arc AQ subtend an angle θ at the centre O of the circle; then AP subtends an angle $m\theta$, and the equation to the chord PQ referred to rectangular axes of which OA is the axis of x is, OA $= a$,

$$x \cos \tfrac{1}{2}(1 \pm m)\theta + y \sin \tfrac{1}{2}(1 \pm m)\theta = a \cos \tfrac{1}{2}(1 \mp m)\theta,$$

upper or lower signs being taken, according as the arcs are measured in the same or in opposite directions.

Taking one equation

$$x \cos \tfrac{1}{2}(1+m)\theta + y \sin \tfrac{1}{2}(1+m)\theta = a \cos \tfrac{1}{2}(1-m)\theta,$$

a point on the locus is given by this; and

$$\tfrac{1}{2}(1+m)\{-x \sin \tfrac{1}{2}(1+m)\theta + y \cos \tfrac{1}{2}(1+m)\theta\}$$
$$= -\tfrac{1}{2}(1-m)a \sin \tfrac{1}{2}(1-m)\theta.$$

These give

$$x(1+m)=a(\cos m\theta+m\cos\theta),$$
$$y(1+m)=a(\sin m\theta+m\sin\theta);$$

which is an epicycloid, as may be readily seen by comparison with the ordinary equations.

Again, taking the lower sign, we have

$$x(1-m)=a(\cos m\theta-m\cos\theta),$$
$$y(1-m)=a(-\sin m\theta-m\sin\theta),$$

giving a hypocycloid.

6. *Find the values of the definite integrals*

$$\int_0^\infty \frac{\cos ax-\cos bx}{x}dx, \qquad \int_0^{\frac{\pi}{2}} \log(\sin\theta)d\theta.$$

These integrals are worked out in Williamson's Integral Calculus, §§ 119, 118.

7. *Evaluate the definite integral*

$$\int_0^a\int_0^x \frac{y^n dx\,dy}{\sqrt{(a-x)(x-y)}}.$$

Let $y=x\sin^2\theta$, then we require

$$\int_0^a\int_0^{\frac{\pi}{2}} \frac{2x^{n+1}\sin^{2n+1}\theta\,d\theta\,dx}{\sqrt{x(a-x)}}$$
$$=\int_0^a \frac{2x^{n+1}dx}{\sqrt{x(a-x)}}\times\int_0^{\frac{\pi}{2}}\sin^{2n+1}\theta\,.\,d\theta.$$

Now let $x=a\sin^2\phi$, we require

$$a_{n+1}\int_0^{\frac{\pi}{2}} 4\sin^{2n+2}\phi\,d\phi\times\int_0^{\frac{\pi}{2}}\sin^{2n+1}\theta\,.\,d\theta$$

$$= 2a^{n+1}\pi \cdot \frac{1.3.5\ldots 2n+1}{2.4.6\ldots 2n+2} \times \frac{2.4.6\ldots 2n}{1.3.5\ldots 2n+1}$$

$$= \frac{\pi a^{n+1}}{n+1}.$$

8. *Determine the entire volume of the surface represented by the equation*

$$(x^2+y^2+z^2)^6 = (a^2x^2+b^2y^2+c^2z^2)^2.$$

In polar coordinates the equation becomes

$$r^3 = \pm(a^3\cos^3\theta\cos^2\phi + b^3\cos^2\theta\sin^2\phi + c^3\sin^2\theta).$$

Its volume is therefore

$$\frac{2}{3}\int_0^{\frac{\pi}{2}}\int_0^{2\pi} r^3\cos\theta\, d\theta\, d\phi$$

$$= \frac{2\pi}{3}\int_0^{\frac{\pi}{2}} \{(a^3+b^3)\cos^2\theta + 2c^3\sin^2\theta\}\cos\theta\, d\theta$$

$$= \frac{4\pi(a^3+b^3+c^3)}{9}.$$

9. *Integrate the differential equation*

$$\frac{dx}{\sqrt{a+bx+cx^2+dx^3+ex^4}} + \frac{dy}{\sqrt{a+by+cy^2+dy^3+ey^4}} = 0.$$

Call $\quad a+bx+cx^2+dx^3+ex^4,\ \mathrm{X},$

and $\quad a+by+cy^2+dy^3+ey^4,\ \mathrm{Y};$

we have then

$$\frac{dx}{\sqrt{\mathrm{X}}} + \frac{dy}{\sqrt{\mathrm{Y}}} = 0,$$

and
$$\frac{dx}{\sqrt{X}}=\frac{-dy}{\sqrt{Y}}=\frac{d(x-y)}{\sqrt{X}+\sqrt{Y}}$$

$$=\frac{d(x+y)}{\sqrt{X}-\sqrt{Y}}=\frac{d(\sqrt{X}-\sqrt{Y})}{\frac{1}{2}\left(\frac{dX}{dx}+\frac{dY}{dy}\right)},$$

all these results coming by elementary properties of ratios.

Proceeding,

$$\frac{(x-y)d(\sqrt{X}-\sqrt{Y})-(\sqrt{X}-\sqrt{Y})d(x-y)}{\frac{1}{2}(x-y)\left(\frac{dX}{dx}+\frac{dY}{dy}\right)-(X-Y)}=\frac{d(x+y)}{\sqrt{X}-\sqrt{Y}};$$

but
$$(x-y)\left(\frac{dX}{dx}+\frac{dY}{dy}\right)-2(X-Y)$$

$$=(x-y)\{2b+2c(x+y)+3d(x^2+y^2)+4e(x^3+y^3)$$
$$-2\{b+c(x+y)+d(x^2+xy+y^2)$$
$$+e(x^3+x^2y+xy^2+y^3)\}\}$$
$$=(x-y)\{d(x-y)^2+2e(x-y)^2(x+y)\}$$
$$=(x-y)^3\{d+2e(x+y)\}.$$

Thus we get easily

$$2d\,.\,\frac{\sqrt{X}-\sqrt{Y}}{x-y}=\frac{x-y}{\sqrt{X}-\sqrt{Y}}\{d+2e(x+y)\}d(x+y),$$

or
$$\left(\frac{\sqrt{X}-\sqrt{Y}}{x-y}\right)^2=c'+d(x+y)+e(x+y)^2$$

which is the integral required.

See Cayley's Elliptic Functions, § 435.

10. *Solve the differential equations*

$$\frac{d^2y}{dx^2}-\frac{2dy}{dx}+y=x^3, \quad x^2\left(\frac{d^2y}{dx^2}+a^2y\right)=6y,$$

$$x^4\frac{d^2z}{dx^2}=a^2\frac{d^2z}{dy^2}.$$

Taking the first equation we have

$$\frac{d^2}{dx^2}(ye^{-x})=x^3e^{-x};$$

$$\therefore \quad ye^{-x}=e^{-x}(x^3+6x^2+18x+24)+ax+b;$$

$$\therefore \quad y \quad =x^3+6x^2+18x+24+(ax+b)e^x.$$

The second equation is worked in Boole's Differential Equations, ch. xvii. § 4, ex. 6.

In the third equation, let $xt=1$; it becomes

$$t\frac{d^2z}{dt^2}+2\frac{dz}{dt}=a^4t\frac{d^2z}{dy^2},$$

or

$$\frac{d^2}{dt^2}(zt)=a^4\frac{d^2}{dy^2}(zt).$$

From this well-known equation (Boole, ch. xv. § 6, ex. 1) we have

$$zt=\phi(y+a^2t)+\psi(y-a^2t),$$

or

$$z=x\phi\left(y+\frac{a^2}{x}\right)+\psi\left(y-\frac{a^2}{x}\right).$$

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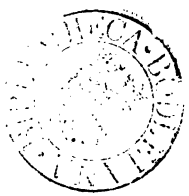

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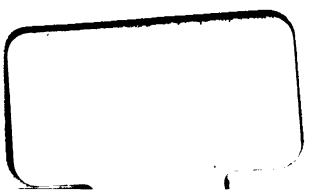
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